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A Characterization of Ternary Rings of Operators*

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0. INTRODUCTION

A *ternary ring of operators* (TRO) between two complex Hilbert spaces H and K is a linear subspace $\mathfrak{R}$ of $\mathscr{L}(H,K)$ satisfying $AB^*C \in \mathfrak{R}$ for all $A, B, C \in \mathfrak{R}$. The purpose of this paper is to give a characterization of a TRO as an abstract ternary ring.

In [2] Hestenes introduced a purely algebraic concept of an abstract ternary ring which was related to the study of TROs between finite dimensional spaces and which was pursued further by other authors, e.g., [10, 11]. Our approach applies to infinite dimensional spaces and is oriented at the representation of a C^*-algebra (respectively, W^*-algebra) as a norm (respectively, weakly) closed star algebra of operators on a Hilbert space. Thus, we confine our considerations to TROs which are norm or weakly closed. The first step towards a characterization of these rings is to find a suitable concept of an abstract ternary ring which reflects the ternary operation, $(A, B, C) \mapsto AB^*C$, the norm relations, and, in the second case, also the weak-operator structure. This leads us to the following basic definition:

0.1. DEFINITION. A *ternary C^*-ring* $(\mathfrak{X}, (\cdot,\cdot,\cdot), \|\cdot\|)$ consists of a complex Banach space $(\mathfrak{X}, \|\cdot\|)$ and a ternary operation $(\cdot,\cdot,\cdot)$: $\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ such that for all $v, w, x, y, z \in \mathfrak{X}$ and $\lambda \in \mathbb{C}$ the following conditions hold:

$$\begin{aligned}(\lambda x + y, v, w) &= \lambda(x, v, w) + (y, v, w),\\ (v, w, \lambda x + y) &= \lambda(v, w, x) + (v, w, y),\end{aligned} \tag{0.1}$$

$$(v, \lambda x + y, w) = \bar{\lambda}(v, x, w) + (v, y, w), \tag{0.2}$$

$$((v, w, x), y, z) = (v, (y, x, w), z) = (v, w, (x, y, z)), \tag{0.3}$$

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$$\|(x, y, z)\| \leqslant \|x\|\,\|y\|\,\|z\|, \tag{0.4}$$

$$\|(x, x, x)\| = \|x\|^3. \tag{0.5}$$

A *ternary W^*-ring* is a ternary C^*-ring which is, in addition, a dual space.

Starting from these definitions we prove the following representation theorems which we give here in shortened versions.

THEOREM 3. *For each ternary C^*-ring $(\mathfrak{X}, (\cdot, \cdot, \cdot), \|\cdot\|)$ there exists one, and only one, operator $T: \mathfrak{X} \to \mathfrak{X}$ satisfying*

(1) $T^2 = I$,

(2) $T((x, y, z)) = (Tx, y, z) = (x, Ty, z) = (x, y, Tz)$ *for all $x, y, z \in \mathfrak{X}$, and*

(3) $(\mathfrak{X}, T \circ (\cdot, \cdot, \cdot), \|\cdot\|)$ *is a ternary C^*-ring which is isomorphic to a norm closed* TRO.

THEOREM 4.1. *For each ternary W^*-ring $(\mathfrak{X}, (\cdot, \cdot, \cdot), \|\cdot\|)$ the operator T is weak* continuous, and $(\mathfrak{X}, T \circ (\cdot, \cdot, \cdot), \|\cdot\|)$ is a ternary W^*-ring which is (isometrically and weakly*) isomorphic to a weakly closed* TRO.

These results are based upon two representation theorems for Hilbert modules over C^*-algebras which say, first, that each such module can be faithfully represented as a norm closed TRO (cf. Theorem 2.6), and, second, that a self-dual Hilbert module over a W^*-algebra is isomorphic to a weakly closed TRO (cf. Theorem 2.8). For the proof in both cases we use a GNS-construction extending ideas found in [5].

The deduction of our main theorems from these results is as follows: We construct a C^*-algebra $\mathfrak{A} \subset \mathscr{L}(\mathfrak{X})$ and a form $\mathfrak{a}: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{A}$ which contain all information on $(\mathfrak{X}, (\cdot, \cdot, \cdot), \|\cdot\|)$, in the sense that

$$(x, y, z) = \mathfrak{a}(z, y)(x) \qquad \text{and} \qquad \|x\|^2 = \|\mathfrak{a}(x, x)\|$$

hold for all $x, y, z \in \mathfrak{X}$ (cf. Proposition 3.2). Moreover, the pair $(\mathfrak{X}, \mathfrak{a})$ satisfies the axioms for a Hilbert $\mathfrak{A}$-module, apart from $\mathfrak{a}$ being positive. If we set $T = 2P - I$, where $P \in \mathscr{L}(\mathfrak{X})$ is the projection with range $\{x \mid \mathfrak{a}(x, x) \geqslant 0\}$ and kernel $\{x \mid \mathfrak{a}(x, x) \leqslant 0\}$, we see that the form $\mathfrak{a}_T: (x, y) \longmapsto \mathfrak{a}(Tx, y)$ is positive and that $(\mathfrak{X}, \mathfrak{a}_T)$ is a Hilbert $\mathfrak{A}$-module on which then 2.6 can be applied (cf. 3.7 infra). In the case of a ternary W^*-ring the double centralizer algebra $M(\mathfrak{A})$ of $\mathfrak{A}$ turns out to be a W^*-algebra, and $(\mathfrak{X}, \mathfrak{a}_T)$ becomes a self-dual Hilbert $M(\mathfrak{A})$-module (cf. Corollary 4.10), so that Theorem 4.1 is a consequence of Theorem 2.8.

We then prove that $M(\mathfrak{A})_*$ is a quotient of $\mathfrak{X} \hat{\otimes}_\gamma \mathfrak{X}_*$ (cf. Corollary 4.11), and show that the predual $\mathfrak{X}_*$ of $\mathfrak{X}$ coincides with $\mathfrak{X}_c \otimes M(\mathfrak{A})_*$ (cf. Example

4.12). Since $\mathfrak{A}$ depends only on $\mathfrak{X}$, the predual is uniquely determined up to isomorphism. Finally, we prove that each weakly closed TRO is a self-dual Hilbert module over a W^*-algebra, and thus a ternary W^*-ring (cf. 4.12).

On the other hand, it is fairly obvious that any norm closed TRO between two Hilbert spaces (such as the space of all compact operators or a C^*-algebra in $\mathscr{L}(H)$) is a ternary C^*-ring. All these examples lead to $T = I$; in general, however, $T \neq I$ (cf. Example 3.8). Hence, the problem is to characterize the case $T = I$. We have no answer to this question, but give two ideas to it at the end of Section 3.

In our considerations we need some fundamental properties of TROs: in particular we need statements concerning partial isometries and a density theorem of Kaplansky type. All this is provided in Section 1.

We make the following conventions: The linear spaces considered here are over the complex field, $\mathbb{C}$. If X is a linear space, then X_c denotes X equipped with the conjugate scalar multiplication $(\lambda, x) \mapsto \bar{\lambda}x$, and we write I for the identity operator on X. If X and Y are normed spaces, then $\mathscr{L}(X, Y)$ denotes the set of all bounded operators from X to Y. The symbol X^* (respectively, X_*) is used for the dual space of X (respectively, predual of X whenever it exists), and $\bar{\Omega}^n$ stands for the norm closure of $\Omega \subset X$. The action of an algebra $\mathfrak{A}$ on a right $\mathfrak{A}$-module X is denoted by $(x, a) \mapsto x \cdot a$, and it is always assumed to satisfy $\lambda(x \cdot a) = (\lambda x) \cdot a = x \cdot (\lambda a)$ for all $\lambda \in \mathbb{C}$, $x \in \mathfrak{X}$, $a \in \mathfrak{A}$. Similarly, for left $\mathfrak{A}$-modules. The reference for Banach $\mathfrak{A}$-modules and approximate identities is [13, Part I, Sect. 2]. The notions and facts on C^*-algebras and W^*-algebras we use can be found in [9].

1. Ternary Rings of Operators: Preliminaries

This section collects some basic material which will subsequently be used in connection with TROs arising from Representation Theorems 2.6 and 3.1.

In the following, $\mathfrak{R}$ denotes a norm closed TRO between two Hilbert spaces H and K. It is clear that the weak-operator closure $\bar{\mathfrak{R}}^\sigma$ of $\mathfrak{R}$ is also a TRO. By $[\mathfrak{R}]_H$ (respectively, $\{\mathfrak{R}\}_H$) we shall denote the C^*-algebra (respectively, W^*-algebra) generated by $\{A^*B \mid A, B \in \mathfrak{R}\}$ in $\mathfrak{L}(H)$. Similarly, we define $[\mathfrak{R}]_K$ and $\{\mathfrak{R}\}_K$.

For instance, the space $\mathfrak{C}(H, K)$ of all compact operators is a norm closed TRO with weak closure $\mathscr{L}(H, K)$, while $[\mathfrak{C}(H, K)]_H$ is equal to $\mathfrak{C}(H)$. When $\mathfrak{A}$ is a C^*-algebra and $\{\pi, H\}$, $\{\pi', K\}$ are two *-representations of $\mathfrak{A}$ (cf. [9, Def. 1.16.3]), the intertwining ring $\mathfrak{J} = \{A \in \mathscr{L}(H, K) \mid A\pi(a) = \pi'(a)A, a \in \mathfrak{A}\}$ is a weakly closed TRO, and $\{\mathfrak{J}\}_H$ coincides with the commutant of $\pi(\mathfrak{A})$ in $\mathscr{L}(H)$.

Our first considerations are related to the partial isometries in $\bar{\mathfrak{R}}^\sigma$. These appear naturally in the polar decomposition of operators (cf.

Proposition 1.2) and provide the extreme points in the unit ball of $\overline{\mathfrak{R}}^{\sigma}$ (cf. Lemma 1.3). The latter fact will lead to a polar decomposition for weak* continuous functionals on a ternary W^*-ring (cf. Lemma 4.4). Further, we shall make essential use of C^*-algebras associated with a single partial isometry. To this end we introduce the following notations which are partly due to [3, Sect. 15]:

1.1. *Notations.* Let $R \in \mathfrak{R}$ be a partial isometry, i.e., $R = RR^*R$, and denote by E and F the support and range projection R^*R and RR^* of R, respectively. Define

$$\mathfrak{R}_{(R)} = \{X \in \mathfrak{R} \mid X = XE = FX\}.$$

Then, endowed with the operator norm, $\mathfrak{R}_{(R)}$ becomes a *C^*-algebra with unit R by*

$$X \circ Y = XR^*Y, \qquad X^{\#} = RX^*R. \tag{1.1}$$

Obviously, $X \mapsto R^*X$ is a *-isomorphism of $\mathfrak{R}_{(R)}$ onto $E[\mathfrak{R}]_H E$.

Next, put $\mathfrak{S} = \overline{\mathfrak{R}}^{\sigma}$, so that $\mathfrak{S}_{(R)} = \overline{\mathfrak{R}}^{\sigma}_{(R)}$. Then $\mathfrak{S}_{(R)}$ is a W^*-algebra since the *-isomorphism $X \mapsto R^*X$ of $\mathfrak{S}_{(R)}$ onto $E\{\mathfrak{R}\}_H E$ is a homeomorphism in the weak-operator topologies. Note that $\mathfrak{R}_{(R)}$ (respectively, $\mathfrak{S}_{(R)}$) is a C^*-subalgebra (respectively, W^*-subalgebra) of $\mathscr{L}(H, K)_{(R)}$.

1.2. PROPOSITION. *Let $X \in \mathfrak{R}$. If R denotes the partial isometry of the polar decomposition $X = R\,|X|$ of X, then the C^*-algebra (respectively, W^*-algebra) generated by X in $\mathscr{L}(H, K)_{(R)}$ is equal to the smallest norm (respectively, weakly) closed* TRO *containing X. In particular, $R \in \overline{\mathfrak{R}}^{\sigma}$.*

Proof. Let $P = \mathbb{C}[t]$ denote the polynomials, define $P_0 = \{p \in P \mid p(0) = 0\}$ and write $p_R(X)$ for the polynomial associated with X and $p \in P$ in $\mathscr{L}(H, K)_{(R)}$. By [3, Theorem 6.2], $|X| = R^*X$. Since $A \mapsto RA$ is the inverse of the *-isomorphism $Y \mapsto R^*Y$ of $\mathscr{L}(H, K)_{(R)}$ onto $R^*R\mathscr{L}(H)R^*R$ we obtain, using the first equation in (1.1) for $\mathscr{L}(H, K)$ instead of $\mathfrak{R}$,

$$Xp(|X|) = X \circ (Rp(|X|)) = X \circ p_R(X) = (tp)_R(X).$$

Hence, the C^*-algebra $\mathfrak{C} = \overline{\{p_R(X) \mid p \in P_0\}}^{n}$ generated by X in $\mathscr{L}(H, K)_{(R)}$ is equal to $\overline{\{Xp(|X|) \mid p \in P\}}^{n}$.

Now if $\mathfrak{T}$ denotes the norm closed TRO generated by X, then $|X| \in [\mathfrak{T}]_H$ and $\mathfrak{T}([\mathfrak{T}]_H \oplus \mathbb{C}) \subset \mathfrak{T}$, which imply $\mathfrak{C} \subset \mathfrak{T}$. On the other hand, by (1.1),

$$p_R(X)\, q_R(X)^* r_R(X) = p_R(X) \circ q_R(X)^{\#} \circ r_R(X) = (p\bar{q}r)_R(X).$$

So, $\mathfrak{C}$ is a norm closed TRO containing X. Thus, $\mathfrak{C} = \mathfrak{T}$.

Similarly, $\overline{\{p_R(X) \mid p \in P_0\}}^{\sigma} = \overline{\mathfrak{T}}^{\sigma}$. The unit S of the W^*-subalgebra $\overline{\mathfrak{T}}^{\sigma}$ of

$\mathscr{L}(H,K)_{(R)}$ is a partial isometry in $\mathscr{L}(H,K)$, because $S = S \circ S^{\#} \circ S = SR^*RS^*RR^*S = SS^*S$. Moreover,

$$SS^*X = SR^*RS^*RR^*X = S \circ S^{\#} \circ X = S \circ X = X$$

and, analogously, $XS^*S = X$. Thus, $SS^* \geqslant RR^*$ and $S^*S \geqslant R^*R$ in $\mathscr{L}(K)$ and $\mathscr{L}(H)$, respectively, and therefore $S = R$. In particular, we have $R \in \bar{\mathfrak{T}}^{\sigma} \subset \bar{\mathfrak{R}}^{\sigma}$.

1.3. Lemma. *For each X in the unit ball $D_1(\mathfrak{R})$ of $\mathfrak{R}$ the following conditions are equivalent*:

(1) *X is an extreme point of $D_1(\mathfrak{R})$;*

(2) *X is a partial isometry satisfying $(I - XX^*)\,\mathfrak{R}(I - X^*X) = \{0\}$;*

(3) *$(I - XX^*)\,\mathfrak{R}(I - X^*X) = \{0\}$.*

Proof. Let E and F denote the support and range projection of X, respectively. Observe that

$$(I - F)\,\mathfrak{R}(I - E) = \{Y \in \mathfrak{R} \mid 0 = Y^*X, 0 = YX^*\}. \tag{1.2}$$

(1) $\Rightarrow$ (2). Let R be the partial isometry of the polar decomposition $X = R\,|X|$ of X. In view of 1.1 and 1.2, $Y \mapsto R^*Y$ is a $*$-isomorphism of the norm closed TRO generated by X onto the C^*-algebra generated by $|X| = R^*X$. Thus, using [9, Proposition 1.6.2], X is a partial isometry. Moreover, by (1.2), for each $Y \in (I - F)\,\mathfrak{R}(I - E)$ with $\|Y\| \leqslant 1$ we have

$$\|Y + X\|^2 = \|(Y + X)^*(Y + X)\| = \|Y^*Y + X^*X\| = \operatorname{Max}\{\|X\|^2, \|Y\|^2\} \leqslant 1$$

which yields $Y = 0$, as desired.

(3) $\Rightarrow$ (1). From $0 = (I - XX^*)\,X(I - X^*X)$ we obtain $0 = X^*X(I - X^*X)$. Hence, X^*X is a projection, and so, X is a partial isometry. Now, fix $Y, Z \in D_1(\mathfrak{R})$ with $X = (Y + Z)/2$. Since FYE and FZE belong to the unit ball of the C^*-algebra $\mathfrak{R}_{(X)}$, the arguments in [9, Theorem 1.6.4] show that $X = FYE = FZE$. From

$$\begin{aligned} 1 \geqslant \|YE\|^2 &= (FYE + (I - F)YE)^*(FYE + (I - F)YE)\| \\ &= \|X + (I - F)YE\|^2 = 1 + \|(1 - F)YE\|^2 \end{aligned}$$

we deduce that $FYE = YE$. Similarly, $FZE = ZE$. Thus, $X = YE = ZE$.

Since (3) implies that $(I - E)\,\mathfrak{R}^*(I - F) = \{0\}$, the same line of reasoning with X^* and $\mathfrak{R}^*$ instead of X and $\mathfrak{R}$ yields $X = FY = FZ$. Therefore,

$$\begin{aligned} Y &= YE + Y(I - E) \\ &= X + FY(I - E) + (I - F)\,Y(I - E) = X \qquad \text{and} \qquad Z = X. \end{aligned}$$

Our next result is an analogue of Kaplansky's density theorem which we shall use for Hilbert modules and ternary W^*-rings. For the proof we modify the arguments in [13, Sect. 4, Theorem 2.5].

1.4. PROPOSITION. *The unit ball of $\mathfrak{R}$ is weakly and strongly dense in the unit ball of the weak closure $\bar{\mathfrak{R}}^\sigma$ of $\mathfrak{R}$.*

Proof. Obviously,

$$\tilde{\mathfrak{R}} = \left\{ \begin{pmatrix} 0 & Y^* \\ X & 0 \end{pmatrix} \middle| X, Y \in \mathfrak{R} \right\}$$

is a norm closed, self-adjoint TRO in $\mathscr{L}(H \oplus K)$, and

$$X \mapsto \begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix}$$

is a real-linear isometry and a weak homeomorphism of $\mathfrak{R}$ onto the self-adjoint part $\tilde{\mathfrak{R}}_{sa}$ of $\tilde{\mathfrak{R}}$. Moreover, observe that the weak and strong operator topologies coincide on convex sets. Thus, if we assume that $H = K$ and that $\mathfrak{R}$ is self-adjoint, it remains to show that $D_1(\mathfrak{B})$ is contained in the strong closure of $D_1(\mathfrak{A})$, where $\mathfrak{A} = \mathfrak{R}_{sa}$, $\mathfrak{B} = (\bar{\mathfrak{R}}^\sigma)_{sa}$. To this end we need

LEMMA 1.4.1. *Let $X \in \mathscr{L}(H)_{sa}$ and let $\mathfrak{T}$ denote the norm closed* TRO *generated by X. Then there is an $X' \in \mathfrak{T}$ satisfying $X = 2X'(1 + X'^2)^{-1}$.*

Proof. By 1.2, $\mathfrak{T}$ is equal to the C^*-algebra generated by X in $\mathscr{L}(H)_{(R)}$. Let $\chi \mapsto \chi_R(X)$ denote its functional calculus. Define the function f on $[0, 1]$ by $f(t) = 2t/(1 + t^2)$, and write $X' = (f^{-1})_R(X)$. Then X' is positive in the C^*-algebra $\mathfrak{T}$, and thus R^*X' is positive in $\mathscr{L}(H)$. Since $X = X^*$ there are two projections E and F with $R = E - F$, $EF = 0$. From $RX = XR$ we conclude that $\mathfrak{T}$ commutes with R in $\mathscr{L}(H)$. Hence, $RX' = X'R$. Moreover, $X' = X'^*$, as $X'R$ is positive. Therefore,

$$X = f_R(X') = Rf(RX') = R(2RX'(1 + R^2X'^2)^{-1}) = 2X'(1 + X'^2)^{-1},$$

which proves Lemma 1.4.1.

Now, fix $X \in D_1(\mathfrak{B})$, write $X = 2X'(1 + X'^2)^{-1}$ with $X' \in \mathfrak{B}$ according to Lemma 1.4.1, and define $Y = 2Y'(1 + Y'^2)^{-1}$, $Y' \in \mathfrak{A}$. Then, for each $Y' \in \mathfrak{A}$ we have $\|Y\| \leqslant 1$, $\|(1 + Y'^2)^{-1}\| \leqslant 1$ and

$$(X - Y)/2 = (1 + Y'^2)^{-1}((X' - Y') + Y'(Y' - X')X')(1 + X'^2)^{-1}.$$

Since $\mathfrak{B}$ is contained in the strong closure of $\mathfrak{A}$ we are done.

The following statement is necessary for the proof of Corollary 4.7:

1.5. LEMMA. *If $\mathfrak{J}$ is a weakly closed two-sided ideal in $\bar{\mathfrak{R}}^\sigma$ (that is, a subspace of $\bar{\mathfrak{R}}^\sigma$ satisfying $\bar{\mathfrak{J}}^\sigma = \mathfrak{J} \supset \mathfrak{J}\mathfrak{R}^*\mathfrak{R} \cup \mathfrak{R}\mathfrak{R}^*\mathfrak{J}$), then there is a central porjection $E \in \{\mathfrak{R}\}_H$ such that $\mathfrak{J} = \bar{\mathfrak{R}}^\sigma E$.*

Proof. Put $\mathfrak{A} = \{\mathfrak{R}\}_H$ and $\mathfrak{B} = \{\mathfrak{J}\}_H$. Then $\mathfrak{J}\mathfrak{A} \subset \mathfrak{J}$. Obviously, $\mathfrak{L} = \{A \in \mathfrak{A} \mid \bar{\mathfrak{R}}^\sigma A \subset \mathfrak{J}\}$ is a weakly closed two-sided ideal in $\mathfrak{A}$. Hence, $\mathfrak{L} = \mathfrak{A}E$ with a central projection $E \in \mathfrak{A}$. From $\mathfrak{R}\mathfrak{J}^*\mathfrak{J} \subset \mathfrak{R}\mathfrak{R}^*\mathfrak{J} \subset \mathfrak{J}$ we obtain $\mathfrak{J}^*\mathfrak{J} \subset \mathfrak{L}$ and so $\mathfrak{B} \subset \mathfrak{L}$. On the other hand, $\bar{\mathfrak{R}}^\sigma \mathfrak{L} \subset \mathfrak{J}$ implies $\mathfrak{J} = \mathfrak{J}\mathfrak{B} \subset \bar{\mathfrak{R}}^\sigma \mathfrak{L} \subset \mathfrak{J}$. Thus, $\mathfrak{J} = \bar{\mathfrak{R}}^\sigma \mathfrak{L}$ and therefore $\mathfrak{J} = \bar{\mathfrak{R}}^\sigma \mathfrak{A}E = \bar{\mathfrak{R}}^\sigma E$.

2. THE REPRESENTATION OF A HILBERT MODULE AS A TERNARY RING OF OPERATORS

In the following, we establish two representation theorems which bring Hilbert modules in relation to TROs, providing the essential tool for Sections 3 and 4.

The fundamental results we are going to use can be found in [5]. Most of the basic material is also contained in [8], where Hilbert $\mathfrak{A}$-modules in the following sense appear as right $\mathfrak{A}$-rigged spaces:

2.1. DEFINITION. Let $\mathfrak{A}$ be a C^*-algebra. A *Hilbert $\mathfrak{A}$-module* $(\mathfrak{H}, \langle \cdot \mid \cdot \rangle)$ consists of a right $\mathfrak{A}$-module $\mathfrak{H}$ and a conjugate bilinear map $\langle \cdot \mid \cdot \rangle: \mathfrak{H} \times \mathfrak{H} \to \mathfrak{A}$, called the *$\mathfrak{A}$-valued inner product*, such that:

(1) $\langle x \mid x \rangle \geqslant 0$ and $\langle x \mid x \rangle = 0$ only if $x = 0$, $x \in \mathfrak{H}$,

(2) $\langle x \cdot a \mid y \rangle = \langle x \mid y \rangle a$, $x, y \in \mathfrak{H}$, $a \in \mathfrak{A}$,

(3) $\langle x \mid y \rangle^* = \langle y \mid x \rangle$, $x, y \in \mathfrak{A}$,

hold, and such that the norm $\|\cdot\|_{\mathfrak{H}}: x \mapsto \|\langle x \mid x \rangle\|^{1/2}$ on $\mathfrak{H}$ is complete (cf. [5, Definitions 2.1 and 2.5]). If, in addition, each $\|\cdot\|_{\mathfrak{H}}$-bounded $\mathfrak{A}$-module homomorphism of $\mathfrak{H}$ into $\mathfrak{A}$ is of the form $y \mapsto \langle y \mid x \rangle$ for some $x \in \mathfrak{H}$, then $(\mathfrak{H}, \langle \cdot \mid \cdot \rangle)$ is called *self-dual over* $\mathfrak{A}$ (cf. [5, paragraph 3]).

For instance, let $\mathfrak{R} \subset \mathscr{L}(H, K)$ be a norm closed TRO and put $\langle X \mid Y \rangle = Y^*X$. Then $(\mathfrak{R}, \langle \cdot \mid \cdot \rangle)$ is obviously a Hilbert $[\mathfrak{R}]_H$-module with norm $\|\cdot\|_{\mathfrak{R}}$ equal to the operator norm. Moreover, as 4.12 will show, $(\mathfrak{R}, \langle \cdot \mid \cdot \rangle)$ is self-dual over $\{\mathfrak{R}\}_H$ if $\mathfrak{R}$ is weakly closed. Conversely, we will prove in Theorem 2.6 that each Hilbert $\mathfrak{A}$-module $(\mathfrak{H}, \langle \cdot \mid \cdot \rangle)$ is isomorphic to $(\mathfrak{R}, \langle \cdot \mid \cdot \rangle)$ for a suitable norm closed TRO $\mathfrak{R}$. The isomorphism is given by an isometry $U: \mathfrak{H} \to \mathfrak{R}$ and a $*$-isomorphism $\pi: \mathfrak{A} \to [\mathfrak{R}]_H$ satisfying

$$U(x \cdot a) = U(x)\,\pi(a), \qquad U(y)^*U(x) = \pi(\langle x \mid y \rangle), \qquad x, y \in \mathfrak{H}, \quad a \in \mathfrak{A}.$$

In particular, U and π result as the restrictions to $\mathfrak{H}$ and $\mathfrak{A}$, respectively, of an analogous representation of the self-dual completion $\mathfrak{H}''$ of $\mathfrak{H}$ over $\mathfrak{A}^{**}$, which also involves the naturally occurring weak and strong topologies. Then, for a W^*-algebra $\mathfrak{A}$ and a self-dual Hilbert $\mathfrak{A}$-module $(\mathfrak{H}, \langle\cdot\mid\cdot\rangle)$, we shall derive in Theorem 2.8 that $(\mathfrak{H}, \langle\cdot\mid\cdot\rangle)$ is isomorphic to $(\mathfrak{R}, \langle\cdot\mid\cdot\rangle)$, where $\mathfrak{R}$ is a weakly closed TRO, the isomorphism being likewise a "normal" one. It is easy to see that these theorems contain the well-known representation for C^*- and W^*-algebras by specializing $\mathfrak{H}=\mathfrak{A}$ and $\langle x\mid y\rangle = y^*x$.

For the remainder we thus fix a C^*-algebra $\mathfrak{A}$ and a Hilbert $\mathfrak{A}$-module $(\mathfrak{H}, \langle\cdot\mid\cdot\rangle)$. First, let us recall the self-dual completion $\mathfrak{H}''$ of $\mathfrak{H}$.

2.2. Proposition ([5, Corollary 4.3]). *Endowed with the conjugate scalar multiplication $(\lambda, \tau)\mapsto \bar{\lambda}\tau$, the space $\mathfrak{H}''$ of all $\|\cdot\|_{\mathfrak{H}}$-bounded $\mathfrak{A}$-module homomorphisms of $\mathfrak{H}$ into the enveloping W^*-algebra $\mathfrak{A}^{**}$ of $\mathfrak{A}$ becomes a right $\mathfrak{A}^{**}$-module by*

$$(\tau\cdot A)(x) = A^*\tau(x),\qquad \tau\in\mathfrak{H}'',\quad A\in\mathfrak{A}^{**},\quad x\in\mathfrak{H}.$$

Moreover, the $\mathfrak{A}$-valued inner product can be extended to an $\mathfrak{A}^{**}$-valued inner product on $\mathfrak{H}''$, again denoted by $\langle\cdot\mid\cdot\rangle$, such that:

(1) *$(\mathfrak{H}'', \langle\cdot\mid\cdot\rangle)$ is a self-dual Hilbert $\mathfrak{A}^{**}$-module,*

(2) *$\|\cdot\|_{\mathfrak{H}''}$ is equal to the operator norm induced by $\mathscr{L}(\mathfrak{H}, \mathfrak{A}^{**})$,*

(3) *$x\mapsto\langle\cdot\mid x\rangle$ is an isomorphism of $(\mathfrak{H}, \langle\cdot\mid\cdot\rangle)$ into $(\mathfrak{H}'', \langle\cdot\mid\cdot\rangle)$,*

(4) *$\tau(x) = \langle x\mid\tau\rangle$, $x\in\mathfrak{H}$, $\tau\in\mathfrak{H}''$,*

x and $\langle\cdot\mid x\rangle$ being identified for each $x\in\mathfrak{H}$.

It is shown in [5, Proposition 3.8] that $\mathfrak{H}''$ is weak* closed in the dual space $\mathscr{L}(\mathfrak{H}, \mathfrak{A}^{**})$ of the projective tensor product $\mathfrak{H}\mathbin{\hat{\otimes}_\gamma}\mathfrak{A}^*$. Thus, $\mathfrak{H}''$ is the dual space of a quotient $\mathfrak{B}$ of $\mathfrak{H}\mathbin{\hat{\otimes}_\gamma}\mathfrak{A}^*$ (in the terminology of [7, Corollary 2.13], $\mathfrak{B}=\mathfrak{H}\otimes_{\mathfrak{A}}\mathfrak{A}^*$). As Example 4.12 together with 2.2(1) will prove, the predual of $\mathfrak{H}''$ is uniquely determined up to isomorphism.

2.3. Definition. Henceforth, we call $\sigma(\mathfrak{H}'', \mathfrak{B})$ the *weak topology of* $\mathfrak{H}''$, whereas the *strong topology of* $\mathfrak{H}''$ will be the topology generated by all seminorms $\tau\mapsto\varphi(\langle\tau\mid\tau\rangle)^{1/2}$, $0\leqslant\varphi\in\mathfrak{A}^*$.

2.4. Lemma. *$\mathfrak{H}$ is strongly dense in $\mathfrak{H}''$.*

This follows immediately from the arguments in [6, Lemma 2.3].

The proof of Lemma 2.5 starts with simultaneous GNS-constructions for $(\mathfrak{A}, \varphi)$ and $(\mathfrak{H}, \varphi\circ\langle\cdot\mid\cdot\rangle)$ given in [5, p. 450].

2.5. LEMMA. *Let $0 \leqslant \varphi \in \mathfrak{A}^*$ and denote the associated cyclic representation of $\mathfrak{A}^{**}$ by $\{\pi_\varphi, \mathfrak{A}_\varphi\}$ (cf. [9, Sect. 1.16]). Then there exist a Hilbert space $\mathfrak{H}_\varphi$ and a linear map U_φ from $\mathfrak{H}''$ into $\mathscr{L}(\mathfrak{A}_\varphi, \mathfrak{H}_\varphi)$ with $\|U_\varphi\| \leqslant 1$ such that*:

(1) $U_\varphi(\tau \cdot A) = U_\varphi(\tau)\, \pi_\varphi(A), \tau \in \mathfrak{H}'', A \in \mathfrak{A}^{**}$,

(2) $U_\varphi(\eta)^* U_\varphi(\tau) = \pi_\varphi(\langle \tau \mid \eta \rangle), \tau, \eta \in \mathfrak{H}''$.

Proof. We can suppose that φ is a state. Write N_φ for the kernel $\{x \in \mathfrak{H} \mid \varphi(\langle x \mid x \rangle) = 0\}$ of the positive Hermitian form $\varphi \circ \langle \cdot \mid \cdot \rangle$, and denote the quotient map $\mathfrak{H} \to \mathfrak{H}/N_\varphi$ by $x \mapsto (x)_\varphi$. Then

$$\langle (x)_\varphi \mid (y)_\varphi \rangle_\varphi = \varphi(\langle x \mid y \rangle), \qquad x, y \in \mathfrak{H}$$

defines a scalar product on $\mathfrak{H}/N_\varphi$ with corresponding Hilbert space $\mathfrak{H}_\varphi$. The same process, applied to $\mathfrak{A}$ and φ, leads to the Hilbert space $\mathfrak{A}_\varphi$ and the quotient map $a \mapsto (a)_\varphi$.

By the remarks preceeding [5, Proposition 3.1], for each $\tau \in \mathfrak{H}''$ there is a unique $(\tau)_\varphi \in \mathfrak{H}_\varphi$ such that

$$\varphi(\tau(x)) = \langle (x)_\varphi \mid (\tau)_\varphi \rangle_\varphi, \qquad x \in \mathfrak{H}.$$

It follows that for each $\tau \in \mathfrak{H}''$ an operator $U_\varphi(\tau) \in \mathscr{L}(\mathfrak{A}_\varphi, \mathfrak{H}_\varphi)$ with norm $\leqslant \|\tau\|$ is given by

$$U_\varphi(\tau)(a)_\varphi = (\tau \cdot a)_\varphi, \qquad a \in \mathfrak{A}. \tag{2.1}$$

Indeed, [5, Theorem 2.8] yields $\varphi(\tau(x)^* \tau(x)) \leqslant \|\tau\|^2 (\langle x \mid x \rangle)$ for each $x \in \mathfrak{H}$ and therefore,

$$\begin{aligned} \|(\tau \cdot a)_\varphi\|_\varphi &= \sup\{|\langle (x)_\varphi \mid (\tau \cdot a)_\varphi \rangle_\varphi| \mid \|(x)_\varphi\|_\varphi \leqslant 1\} \\ &= \sup\{|\varphi(a^* \tau(x))| \mid \|(x)_\varphi\|_\varphi \leqslant 1\} \\ &\leqslant \sup\{\varphi(a^*a)^{1/2} \varphi(\tau(x)^* \tau(x))^{1/2} \mid \varphi(\langle x \mid x \rangle) \leqslant 1\} \\ &\leqslant \varphi(a^*a)^{1/2} \|\tau\| = \|(a)_\varphi\|_\varphi \|\tau\|, \end{aligned}$$

so that $U_\varphi(\tau)$ is well defined and continuous on a dense subset of $\mathfrak{A}_\varphi$.

Now, (1) results from Eq. (2.1) and the continuity of $U_\varphi(\tau)$, because

$$\begin{aligned} U_\varphi(\tau \cdot A)(a)_\varphi &= ((\tau \cdot A) \cdot a)_\varphi = (\tau \cdot (Aa))_\varphi \\ &= U_\varphi(\tau)(Aa)_\varphi = U_\varphi(\tau)\, \pi_\varphi(A)(a)_\varphi, \qquad a \in \mathfrak{A}. \end{aligned}$$

It remains to show (2). By Lemma 2.4, there is a net $\{x^\alpha\}_\alpha \subset \mathfrak{H}$ which tends to $\tau \in \mathfrak{H}''$ strongly. Since [5, Proposition 2.3] applied to $(\mathfrak{H}'', \langle \cdot \mid \cdot \rangle)$ gives $\langle \rho \mid \eta \rangle^* \langle \rho \mid \eta \rangle \leqslant \|\eta\|^2 \langle \rho \mid \rho \rangle$ for all $\rho, \eta \in \mathfrak{H}''$, we obtain

$$\|(\langle\tau \cdot a - x^\alpha \cdot a \mid \eta\rangle)_\varphi\|_\varphi^2 = \varphi(\langle\tau \cdot a - x^\alpha \cdot a \mid \eta\rangle^* \langle\tau \cdot a - x^\alpha \cdot a \mid \eta\rangle)$$
$$\leqslant \|\eta\|^2 \varphi(a^*\langle\tau - x^\alpha \mid \tau - x^\alpha\rangle a)$$
$$= \|\eta\|^2 \, \|(\tau \cdot a - x^\alpha \cdot a)_\varphi\|_\varphi \underset{\alpha}{\to} 0.$$

Consequently, for each $a \in \mathfrak{A}$

$$\pi_\varphi(\langle\tau \mid \eta\rangle)(a)_\varphi = (\langle\tau \cdot a \mid \eta\rangle)_\varphi = \lim_\alpha \, (\langle x^\alpha \cdot a \mid \eta\rangle)_\varphi$$
$$= \lim_\alpha U_\varphi(\eta)^* U_\varphi(x^\alpha)(a)_\varphi = U_\varphi(\eta)^* U_\varphi(\tau)(a)_\varphi.$$

Keeping the notations of Lemma 2.5, we put $\Omega = \{\varphi \in \mathfrak{A}^* \mid 0 \leqslant \varphi\}$ and define in turn H, K, π, U to be the l^2-direct sum of $\{\mathfrak{A}_\varphi\}_{\varphi \in \Omega}$, $\{\mathfrak{H}_\varphi\}$, $\{\pi_\varphi\}$, $\{U_\varphi\}$ (note that $\{\pi, H\}$ is the universal *-representation of $\mathfrak{A}$ extended to $\mathfrak{A}^{**}$; cf. [9, Sect. 1.16]). Then we have

2.6. THEOREM. *The map* $U: \mathfrak{H}'' \to \mathscr{L}(H, K)$ *is a linear isometry with the following properties*:

(1) $U(\tau \cdot A) = U(\tau)\, \pi(A), \tau \in \mathfrak{H}'', A \in \mathfrak{A}^{**}$,

(2) $U(\eta)^* U(\tau) = \pi(\langle\tau \mid \eta\rangle),\ \tau, \eta \in \mathfrak{H}''$,

(3) *U is homeomorphic in the strong (respectively, weak) topology of $\mathfrak{H}''$ and the strong (respectively, weak) operator topology*;

(4) *$U(\mathfrak{H})$ is a norm closed* TRO *with weak closure $U(\mathfrak{H}'')$*.

Proof. (1) and (2) are clear from 2.5(1), (2) and the definition of U and π. Since π is isometric, (2) implies

$$\|U(\tau)\|^2 = \|\pi(\langle\tau \mid \tau\rangle)\| = \|\tau\|^2, \qquad \tau \in \mathfrak{H}''.$$

Next, for each net $\{\tau^\alpha\}_\alpha \subset \mathfrak{H}''$ the following conditions are equivalent:

$$\begin{array}{rll} \tau^\alpha \underset{\alpha}{\to} 0 & \text{strongly in} & \mathfrak{H}'', \\ \lim_\alpha \varphi(\langle\tau^\alpha \mid \tau^\alpha\rangle) = 0 & \text{for all} & \varphi \in \Omega, \\ \langle\tau^\alpha \mid \tau^\alpha\rangle \underset{\alpha}{\to} 0 & \text{weak* in} & \mathfrak{A}^{**}, \\ U(\tau^\alpha)^* U(\tau^\alpha) \underset{\alpha}{\to} 0 & \text{weakly in} & \mathscr{L}(H), \\ U(\tau^\alpha) \underset{\alpha}{\to} 0 & \text{strongly in} & \mathscr{L}(H, K). \end{array}$$

Hence, U is a strong homeomorphism.

Thus, in view of Lemma 2.4, $U(\mathfrak{H}'')$ is contained in the strong closure of $U(\mathfrak{H})$. On the other hand, $U(\mathfrak{H}'')$ is weakly closed. Indeed, suppose X belongs to the weak closure of $U(\mathfrak{H}'')$. Then, $X^*U(\mathfrak{H}'') \subset \pi(\mathfrak{A}^{**})$. Since $\tau \mapsto \pi^{-1}(X^*U(\tau))$ is an $\mathfrak{A}^{**}$-module homomorphism of $\mathfrak{H}''$ into $\mathfrak{A}^{**}$, the self-duality of $\mathfrak{H}''$ implies the existence of $\eta \in \mathfrak{H}''$ satisfying $\langle \tau \mid \eta \rangle = \pi^{-1}(X^*U(\tau))$, that is $U(\eta)^*U(\tau) = X^*U(\tau)$, $\tau \in \mathfrak{H}''$. Hence, $X = U(\eta) \in U(\mathfrak{H}'')$. Therefore, $U(\mathfrak{H}'')$ is the weak-operator closure of $U(\mathfrak{H})$.

It remains to show that U is a weak homeomorphism. To this end we require two lemmas.

LEMMA 2.6.1. *U^{-1} is weakly continuous.*

Proof. Fix $f \in \mathfrak{B}$ and write $f = f_1 + f_2 + i(f_3 + f_4)$, each f_k being of the form $f_k = \sum_{n \in \mathbb{N}} x_{n,k} \otimes \varphi_{n,k}$ with certain sequences $\{x_{n,k}\}_n \subset \mathfrak{H}$, $\{\varphi_{n,k}\}_n \subset \Omega$ satisfying $\sum_n \|x_{n,k}\|_{\mathfrak{H}} \|\varphi_{n,k}\| < \infty$. Now, fix k and define $y_n = x_{n,k}$, $\psi_n = \varphi_{n,k}$, $n \in \mathbb{N}$. Then the functional $X \mapsto \sum_{n \in \mathbb{N}} \langle X(I)_{\psi_n} \mid (y_n)_{\psi_n} \rangle_{\psi_n}$ is weak-operator continuous, because

$$\sum_{n \geqslant N} |\langle X(I)_{\psi_n} \mid (y_n)_{\psi_n} \rangle_{\psi_n}| \leqslant \|X\| \sum_{n \geqslant N} \|\psi_n\| \, \|y_n\|_{\mathfrak{H}}$$

and the latter tends to zero as $N \to \infty$. Since for each $\tau \in \mathfrak{H}''$

$$\sum_n \langle U(\tau)(I)_{\psi_n} \mid (y_n)_{\psi_n} \rangle_{\psi_n} = \sum_n \psi_n(\langle \tau \mid y_n \rangle) = f_k(\tau)$$

holds, the assertion follows.

LEMMA 2.6.2. *A linear functional f on $\mathfrak{H}''$ is weakly continuous if, and only if, it is strongly continuous on $D_1(\mathfrak{H}'')$.*

Proof. First, let $\Lambda \subset \mathfrak{H}''$ be convex and strongly closed. Since U is a strong homeomorphism, $U(\Lambda)$ is strongly, and hence weakly, closed in $\mathscr{L}(H, K)$. By Lemma 2.6.1, Λ is weakly closed. Next, suppose f is strongly continuous on $D_1(\mathfrak{H}'')$. Then $\{\tau \in D_1(\mathfrak{H}'') \mid f(\tau) = 0\}$ is convex and weakly closed and so, f is weakly continuous.

The weak continuity of U now follows from 2.6.2 and the observation that U is weakly continuous on norm bounded sets, which in turn is immediate from Eq. (2.1) and the definition of $\mathfrak{B}$:

$$\langle (x)_\varphi \mid U(\tau)(a)_\varphi \rangle_K = \langle (x)_\varphi \mid U_\varphi(\tau)(a)_\varphi \rangle_\varphi = \varphi(a^*\tau(x)),$$
$$x \in \mathfrak{B}, \quad a \in \mathfrak{A}, \quad \tau \in \mathfrak{H}'', \quad \text{and} \quad \varphi \in \Omega.$$

2.7. COROLLARY. *The unit ball of $\mathfrak{H}''$ is both weakly and strongly dense in the unit ball of $\mathfrak{H}''$.*

Proof. By 1.4, $D_1(U(\mathfrak{H}))$ is both weakly and strongly dense in $D_1(U(\mathfrak{H}''))$. The assertion now follows from 2.6(3).

We conclude this section with the analogue of Theorem 2.6 for self-dual Hilbert modules over W^*-algebras. Since the proof is very similar to that of 2.6, we omit the details.

2.8. THEOREM. *Suppose $\mathfrak{A}$ is a W^*-algebra and $(\mathfrak{H}, \langle \cdot \mid \cdot \rangle)$ is a self-dual Hilbert $\mathfrak{A}$-module. Put $\Lambda = \{\varphi \in \mathfrak{A}_* \mid 0 \leqslant \varphi\}$ and, in the notations of Lemma 2.5, define H, K, π, U to be the l^2-direct sum of $\{\mathfrak{A}_\varphi\}_{\varphi \in \Lambda}$, $\{\mathfrak{H}_\varphi\}$, $\{\pi_\varphi \mid \mathfrak{A}\}$, $\{U_\varphi \mid \mathfrak{H}\}$, respectively. Then $U\colon \mathfrak{H} \to \mathscr{L}(H, K)$ is a linear isometry, and*

(1) $U(x \cdot a) = U(x)\, \pi(a)$, $x \in \mathfrak{H}$, $a \in \mathfrak{A}$;

(2) $U(y)^* U(x) = \pi(\langle x \mid y \rangle)$, $x, y \in \mathfrak{H}$;

(3) *U is a homeomorphism in the topology generated by all seminorms $x \mapsto \varphi(\langle x \mid x \rangle)^{1/2}$, $\varphi \in \Lambda$ (the topology generated by $\mathfrak{H} \hat{\otimes}_\gamma \mathfrak{A}_*$, respectively) and the strong-operator topology (the weak-operator topology, respectively);*

(4) *$U(\mathfrak{H})$ is a weakly closed* TRO.

3. THE REPRESENTATION THEOREM FOR TERNARY C^*-RINGS

A ternary C^*-ring $(\mathfrak{X}, (\cdot, \cdot, \cdot), \|\cdot\|)$ will be called *isomorphic* (respectively, *anti-isomorphic*) *to a* TRO, if there are Hilbert spaces H and K and a linear isometry $U\colon \mathfrak{X} \to \mathscr{L}(H, K)$ satisfying

$$U((u, y, z)) = U(x)\, U(y)^* U(z), \qquad x, y, z \in \mathfrak{X}$$

(respectively, $U((x, y, z)) = -U(x)\, U(y)^* U(z)$, $x, y, z \in \mathfrak{X}$). Obviously, $U(\mathfrak{X})$ is a norm closed TRO. The purpose of this section is now the proof of

3.1. REPRESENTATION THEOREM. *Let $(\mathfrak{X}, (\cdot, \cdot, \cdot), \|\cdot\|)$ be a ternary C^*-ring.*

(1) *Then $\mathfrak{X}$ is the direct sum of two ternary C^*-subrings $\mathfrak{X}_+$ and $\mathfrak{X}_-$ in such a way that $\mathfrak{X}_+$ (respectively, $\mathfrak{X}_-$) is isomorphic (respectively, anti-isomorphic) to a* TRO. *Moreover, this decomposition of $\mathfrak{X}$ is unique.*

(2) *If P denotes the projection of $\mathfrak{X}$ onto $\mathfrak{X}_+$ with kernel $\mathfrak{X}_-$, then $2P - 1$ is the only operator $T \in \mathscr{L}(\mathfrak{X})$ which satisfies $T^2 = 1$ and $T((x, y, z)) = (Tx, y, z) = (x, Ty, z) = (x, y, Tz)$, $x, y, z \in \mathfrak{X}$ and which has the property that $(\mathfrak{X}, T \circ (\cdot, \cdot, \cdot), \|\cdot\|)$ is isomorphic to a* TRO.

The proof is carried out in several steps (3.2–3.7). The first one shows that a ternary C^*-ring is a special module over a C^*-algebra:

3.2. PROPOSITION. *Let* $(\mathfrak{X}, (\cdot,\cdot,\cdot), \|\cdot\|)$ *be a ternary* C^**-ring. Then there exists a unique pair* $(\mathfrak{A}, \mathfrak{a})$ *such that*

(1) $\mathfrak{A}$ *is a* C^**-algebra, and* $\mathfrak{X}$ *is a right Banach* $\mathfrak{A}$*-module,*

(2) $\mathfrak{a}\colon \mathfrak{X} \times \mathfrak{X} \to \mathfrak{A}$ *is conjugate bilinear with* $\|\mathfrak{a}\| \leqslant 1$ *and*

$$\mathfrak{a}(x \cdot a, y) = \mathfrak{a}(x, y)a, \qquad x, y \in \mathfrak{X}, \quad a \in \mathfrak{A},$$

$$\mathfrak{a}(x, y)^* = \mathfrak{a}(y, x), \qquad x, y \in \mathfrak{X},$$

(3) $(x, y, z) = x \cdot \mathfrak{a}(z, y), x, y, z \in \mathfrak{X}$,

(4) *the linear hull of* $\mathfrak{a}(\mathfrak{X}, \mathfrak{X})$ *is norm dense in* $\mathfrak{A}$.

Moreover, we have $\|x\|^2 = \|\mathfrak{a}(x, x)\|$ *for each* $x \in \mathfrak{X}$.

Proof. The last statement is a consequence of Eq. (0.5), condition 3.2(3), and the fact that $\|\mathfrak{a}\| \leqslant 1$:

$$\|x\|^3 = \|x \cdot \mathfrak{a}(x, x)\| \leqslant \|x\| \, \|\mathfrak{a}(x, x)\| \leqslant \|x\|^3, \qquad x \in \mathfrak{X}.$$

Existence. To begin with, let V denote the linear hull of $\{(\cdot, x, y) \mid x, y \in \mathfrak{X}\}$ in $\mathscr{L}(\mathfrak{X})$; observe that $(\cdot, x, y) \in \mathscr{L}(\mathfrak{X})$ by (0.1) and (0.4). We show that V is a pre-C^*-algebra in the operator norm.

From (0.3) it is apparent that

$$(\cdot, x_2, y_2) \circ (\cdot, x_1, y_1) = (\cdot, (x_2, y_1, x_1), y_2)$$

and thus, V is a subalgebra of $\mathscr{L}(\mathfrak{X})$. Furthermore,

$$\mathfrak{j}\left(\sum_{i=1}^{n} (\cdot, x_i, y_i)\right) = \sum_{i=1}^{n} (\cdot, y_i, x_i)$$

defines an involution $\mathfrak{j}$ on V. In fact, if $\sum_i (\cdot, x_i, y_i)$ vanishes on $\mathfrak{X}$, set $A = \sum_i (\cdot, y_i, x_i)$ and use (0.2), (0.3) to obtain

$$\begin{aligned}(Az, Az, Az) &= \sum_{i,j,k} ((z, y_i, x_i), (z, y_j, x_j), (z, y_k, x_k)) \\ &= \sum_{i,j,k} (z, ((z, y_j, x_j), x_i, y_i), (z, y_k, x_k)) \\ &= \sum_{j,k} \left(z, \sum_i ((z, y_j, x_j), x_i, y_i), (z, y_k, x_k)\right) = 0, \qquad z \in \mathfrak{X}\end{aligned}$$

so that $\mathfrak{j}$ is well defined. Clearly, $\mathfrak{j}$ is conjugate linear and satisfies $\mathfrak{j}^2 = 1$, while with $A_i = (\cdot, x_i, y_i)$, $i = 1, 2$ we have

$$\begin{aligned}\mathfrak{j}(A_2 \circ A_1)z &= \mathfrak{j}((\cdot, (x_2, y_1, x_1), y_2))z = (z, y_2, (x_2, y_1, x_1)) \\ &= ((z, y_2, x_2), y_1, x_1) = \mathfrak{j}(A_1) \circ \mathfrak{j}(A_2)z, \qquad z \in \mathfrak{X}\end{aligned}$$

and so, $\mathfrak{j}$ is anti-multiplicative. Finally, relative to "$\circ$" and $\mathfrak{j}$, the norm induced by $\mathscr{L}(\mathfrak{X})$ on V has the C^*-property, for, if we put $A = \sum_{i=1}^{n}(\cdot, x_i, y_i)$, then, by (0.5) and (0.4),

$$\begin{aligned}\|Az\|^3 &= \|(Az, Az, Az)\| = \left\|\sum_i ((z, x_i, y_i), Az, Az)\right\| \\ &= \left\|\sum_i (z, (Az, y_i, x_i), Az)\right\| = \|(z, \mathfrak{j}(A) \circ Az, Az)\| \\ &\leqslant \|z\| \, \|\mathfrak{j}(A) \circ Az\| \, \|Az\| \leqslant \|z\|^2 \, \|\mathfrak{j}(A) \circ A\| \, \|Az\|, \qquad z \in \mathfrak{X},\end{aligned}$$

which implies that $\|A\|^2 \leqslant \|\mathfrak{j}(A) \circ A\|$.

Next, define $\mathfrak{A}$ to be the opposite algebra of the norm closure of V in $\mathscr{L}(\mathfrak{X})$ and denote the multiplication and involution on $\mathfrak{A}$ by $(a, b) \mapsto ab$ and $a \mapsto a^*$, respectively. Then $\mathfrak{A}$ is a C^*-algebra, and $\mathfrak{X}$ becomes a faithful right Banach $\mathfrak{A}$-module by $x \cdot a = a(x)$. If we now introduce $\mathfrak{a}: \mathfrak{X} \times \mathfrak{X} \mapsto \mathfrak{A}$ according to

$$\mathfrak{a}(z, y) = (\cdot, y, z), \qquad y, z \in \mathfrak{X}, \tag{3.1}$$

then (3) and (4) are clear by definition, while, obviously, $\mathfrak{a}$ is conjugate linear with $\|\mathfrak{a}\| \leqslant 1$ and $\mathfrak{a}(x, y)^* = \mathfrak{a}(y, x)$. Moreover, writing $a = \mathfrak{a}(x_1, y_1)$, we conclude that for all $x, y, z \in \mathfrak{X}$

$$x \cdot \mathfrak{a}(y \cdot a, z) = (x, z, (y, y_1, x_1)) = ((x, z, y), y_1, x_1) = x \cdot \mathfrak{a}(y, z)a,$$

hence, $\mathfrak{a}(y \cdot a, z) = \mathfrak{a}(y, z)a$. It follows that $\mathfrak{a}(y \cdot a, z) = \mathfrak{a}(y, z)a$ for all $y, z \in \mathfrak{X}$ and $a \in \mathfrak{A}$.

Uniqueness. Suppose, $(\mathfrak{B}, \mathfrak{b})$ is another pair satisfying (1)–(4) for $\mathfrak{X}$. First, the module action of $\mathfrak{B}$ on $\mathfrak{X}$ gives rise to an anti-homomorphism $m: \mathfrak{B} \to \mathscr{L}(\mathfrak{X})$. By (3) and (4), $m(\mathfrak{b}(x, y)) = \mathfrak{a}(x, y)$. Furthermore, m is a $*$-homomorphism, because $m(\mathfrak{b}(x, y)^*) = \mathfrak{a}(y, x) = \mathfrak{a}(x, y)^* = m(\mathfrak{b}(x, y))^*$. Finally,

$$m(\mathfrak{B}) = m(\overline{\operatorname{lin} \mathfrak{b}(\mathfrak{X}, \mathfrak{X})^{n}}) \subset \overline{\operatorname{lin} m(\mathfrak{b}(\mathfrak{X}, \mathfrak{X}))}^{\,n} = \mathfrak{A}.$$

Thus, m is a $*$-isomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$ which transforms $\mathfrak{b}$ into $\mathfrak{a}$.

With the notations of 3.2, we are now going to show that

$$\mathfrak{X}_+ = \{x \in \mathfrak{X} \mid \mathfrak{a}(x, x) \geqslant 0\} \qquad \text{and} \qquad \mathfrak{X}_- = \{x \in \mathfrak{X} \mid \mathfrak{a}(x, x) \leqslant 0\}$$

define two $\mathfrak{a}$-orthogonal $\mathfrak{A}$-submodules which decompose $\mathfrak{X}$. Besides orthogonality the crucial point consists in proving that $\mathfrak{X}_+$ and $\mathfrak{X}_-$ are linear subspaces of $\mathfrak{X}$. To this end we consider not only the "right" structure of $(\cdot, \cdot, \cdot)$, as we did in Proposition 3.2, but also the "left" one which is provided by the C^*-algebra of all $\mathfrak{a}$-adjoinable operators on $\mathfrak{X}$ and another form $\mathfrak{b}$ (cf. Proposition 3.4). For positive $\mathfrak{a}$, this C^*-algebra has already been studied in [5, 6, 8].

3.3. DEFINITION. Suppose F and G are two maps on $\mathfrak{X}$ satisfying $\mathfrak{a}(Fx, y) = \mathfrak{a}(x, Gy)$ for all $x, y \in \mathfrak{X}$. By 3.2, both F and G are continuous $\mathfrak{A}$-module homomorphisms which determine each other uniquely. Hence, we call G *the* $\mathfrak{a}$*-adjoint* F^* *of* F.

By $\mathfrak{A}_\mathfrak{a}(\mathfrak{X})$ we shall denote the $*$-algebra of all maps F on $\mathfrak{X}$ which possess an $\mathfrak{a}$-adjoint. Using 3.2 again, it follows immediately that $\mathfrak{A}_\mathfrak{a}(\mathfrak{X})$ is a C^*-algebra with unit $I = I_\mathfrak{X}$ in the norm inherited from $\mathscr{L}(\mathfrak{X})$. Observe that $\mathfrak{X}$ is a left Banach $\mathfrak{A}_\mathfrak{a}(\mathfrak{X})$-module.

3.4. PROPOSITION. *Define* $\mathfrak{b}\colon \mathfrak{X} \times \mathfrak{X} \to \mathfrak{A}_\mathfrak{a}(\mathfrak{X})$ *by* $\mathfrak{b}(x, y) \cdot z = x \cdot \mathfrak{a}(z, y)$, $x, y \in \mathfrak{X}$ (*that is*, $\mathfrak{b}(x, y) = (x, y, \cdot)$). *Then*

(1) $\mathfrak{b}$ *is conjugate bilinear with* $\|\mathfrak{b}\| \leqslant 1$ *and* $\mathfrak{b}(F \cdot x, y) = F\mathfrak{b}(x, y)$, $\mathfrak{b}(x, y)^* = \mathfrak{b}(y, x)$, $\|x\|^2 = \|\mathfrak{b}(x, x)\|$, $x, y \in \mathfrak{X}$, $F \in \mathfrak{A}_\mathfrak{a}(\mathfrak{X})$.

(2) *If* $\mathfrak{B}$ *denotes the* C^**-algebra generated by* $\mathfrak{b}(\mathfrak{X}, \mathfrak{X})$, *then* $\mathfrak{A}_\mathfrak{a}(\mathfrak{X})$ *is the double centralizer algebra* $M(\mathfrak{B})$ *of* $\mathfrak{B}$.

Further, let $\mathfrak{A}_\mathfrak{b}(\mathfrak{X})$ *denote the* C^**-algebra of all maps on* $\mathfrak{X}$ *which possess a* $\mathfrak{b}$*-adjoint. Then*

(3) $\mathfrak{A}_\mathfrak{b}(\mathfrak{X})$ *is the double centralizer algebra of the opposite algebra* $\mathfrak{A}_\sim$ *of* $\mathfrak{A}$, *and the form* $\mathfrak{c}\colon \mathfrak{X} \times \mathfrak{X} \to \mathfrak{A}_\mathfrak{b}(\mathfrak{X})$ *constructed for* $\mathfrak{b}$ *in analogy with the process* $\mathfrak{a} \rightsquigarrow \mathfrak{b}$ *coincides with* $\mathfrak{a}$.

(4) $M(\mathfrak{B})$ *commutes with* $M(\mathfrak{A}_\sim)$.

Proof. Since, by Eq. (3.1),

$$\begin{aligned}\mathfrak{a}(\mathfrak{b}(x, y) \cdot z_1, z_2) &= \mathfrak{a}((x, y, z_1), z_2) = (\cdot, z_2, (x, y, z_1)) \\ &= (\cdot, (y, x, z_2), z_1) = \mathfrak{a}(z_1, \mathfrak{b}(y, x) \cdot z_2), \quad x, y, z_1, z_2 \in \mathfrak{X},\end{aligned}$$

it follows that $\mathfrak{b}(x, y) \in \mathfrak{A}_\mathfrak{a}(\mathfrak{X})$ and $\mathfrak{b}(x, y)^* = \mathfrak{b}(y, x)$. For each continuous $\mathfrak{A}$-module homomorphism $F\colon \mathfrak{X} \to \mathfrak{X}$ we obtain

$$\mathfrak{b}(F(x), y) \cdot z = F(x) \cdot \mathfrak{a}(z, y) = F(x \cdot \mathfrak{a}(z, y)) = F(\mathfrak{b}(x, y) \cdot z), \qquad x, y, z \in \mathfrak{X}.$$

In particular, $\mathfrak{B}$ is a two-sided ideal in $\mathfrak{A}_\mathfrak{a}(\mathfrak{X})$.

As (1) is clear, we turn to the form $\mathfrak{c}$ for the system $(\mathfrak{X}, \mathfrak{B}, \mathfrak{b})$. From $\mathfrak{c}(x, y)(z) = \mathfrak{b}(z, y) \cdot x = z \cdot \mathfrak{a}(x, y) = \mathfrak{a}(x, y)(z)$ we conclude that $\mathfrak{c} = \mathfrak{a}$. Thus, the C^*-algebra generated by $\mathfrak{c}(\mathfrak{X}, \mathfrak{X})$ is the anti-homomorphic image $\mathfrak{A}_\sim$ of $\mathfrak{A}$ in $\mathscr{L}(\mathfrak{X})$. As above, $\mathfrak{A}_\sim$ is a two-sided ideal in $\mathfrak{A}_\mathfrak{b}(\mathfrak{X})$. Now, put $\mathfrak{A}_\mathfrak{b} = \mathfrak{A}_\mathfrak{b}(\mathfrak{X})$. In view of [1, Proposition 3.7], we may identify $M(\mathfrak{A}_\sim)$ with $\mathfrak{A}_\mathfrak{b}$

if the annihilator of $\mathfrak{A}_\sim$ in $\mathfrak{A}_\mathfrak{b}$ is trivial and $\mathfrak{A}_\mathfrak{b}$ is complete in the strict $\mathfrak{A}_\sim$-topology.

So, fix $F \in \mathfrak{A}_\mathfrak{b}$ with $F\mathfrak{A}_\sim = \{0\}$. Then, $F(x \cdot a) = 0$ for all $x \in \mathfrak{X}$ and $a \in \mathfrak{A}$. Since, by Cohen's factorization theorem [13, p. 80], we have $\mathfrak{X} = \mathfrak{X} \cdot \mathfrak{A}$ it follows that $F = 0$. Next, let $\{F_\alpha\}_\alpha \subset \mathfrak{A}_\mathfrak{b}$ be a Cauchy net in the strict $\mathfrak{A}_\sim$-topology, which means that

$$0 = \lim_{\alpha,\alpha'} \|(F_\alpha - F_{\alpha'}) \cdot a\| = \lim_{\alpha,\alpha'} \|a \cdot (F_\alpha - F_{\alpha'})\|, \qquad a \in \mathfrak{A}.$$

Using Cohen's theorem again we obtain $0 = \lim_{\alpha,\alpha'} \|(F_\alpha - F_{\alpha'})(x)\|$, $x \in \mathfrak{X}$. Similarly, $\lim_\alpha F_\alpha^*(x)$ exists for all $x \in \mathfrak{X}$, because

$$\begin{aligned} \|(F_\alpha^* - F_{\alpha'}^*)(x \cdot a)\| &= \sup\{\|\mathfrak{b}((F_\alpha^* - F_{\alpha'}^*)(x \cdot a), y)\| \mid y \in D_1(\mathfrak{X})\} \\ &= \sup\{\|\mathfrak{b}(x \cdot a, (F_\alpha - F_{\alpha'})(y))\| \mid y \in D_1(\mathfrak{X})\} \\ &\leqslant \|x\| \sup\{\|(F_\alpha - F_{\alpha'})(y) \cdot a^*\| \mid y \in D_1(\mathfrak{X})\} \\ &= \|x\| \, \|a^*(F_\alpha - F_{\alpha'})\| \xrightarrow[\alpha,\alpha']{} 0. \end{aligned}$$

Now, the operator $F: \mathfrak{X} \to \mathfrak{X}$, defined by $F(x) = \lim_\alpha F_\alpha(x)$ is linear, while $\mathfrak{b}(F(x), y) = \lim_\alpha \mathfrak{b}(F_\alpha(x), y) = \mathfrak{b}(x, \lim_\alpha F_\alpha^*(y))$ yields $F \in \mathfrak{A}_\mathfrak{b}$. Finally, $\{F_\alpha\}_\alpha$ tends strictly to F, because

$$F \cdot a(x) = F(x \cdot a) = \lim_\alpha F_\alpha(x \cdot a) = \left(\lim_a F_\alpha \cdot a\right)(x), \qquad x \in \mathfrak{X}, a \in \mathfrak{A}.$$

The proof of $M(\mathfrak{B}) = \mathfrak{A}_\mathfrak{a}(\mathfrak{X})$ is similar. It remains to show (4). So, take $F = F^* \in \mathfrak{A}_\mathfrak{a}(\mathfrak{X})$. Then $FG = GF$ follows from

$$\begin{aligned} z \cdot \mathfrak{a}(FG(x), y) &= z \cdot \mathfrak{a}(G(x), F(y)) = \mathfrak{b}(z, F(y)) \cdot G(x) \\ &= G(\mathfrak{b}(z, F(y)) \cdot x) = G(z \cdot \mathfrak{a}(F(x), y)) = G(\mathfrak{b}(z, y) \cdot F(x)) \\ &= \mathfrak{b}(z, y) \cdot GF(x) = z \cdot \mathfrak{a}(GF(x), y). \end{aligned}$$

3.5. LEMMA. *Let $x \in \mathfrak{X}$, and let $\mathfrak{T}$ denote the ternary C^*-ring generated by x. There exist $u, v \in \mathfrak{T}$ such that $x = u + v$, $\mathfrak{a}(u, v) = 0$, $\mathfrak{a}(u, u) = \mathfrak{a}(x, x)_+$ and $\mathfrak{a}(v, v) = -\mathfrak{a}(x, x)_-$.*

Proof. We may assume that $\|x\| \leqslant 1$. Put $a = \mathfrak{a}(x, x)$. Then $x_n = x \cdot a_+^{1/n}$ is in $\mathfrak{T}$ for all $n \in \mathbb{N}$. Moreover,

$$\|x_n - x_m\|^2 = \|\mathfrak{a}(x_n - x_m, x_n - x_m)\| = \|a_+^{1/2}(a_+^{1/n} - a_+^{1/m})\|^2.$$

Now let $\mathfrak{C}$ denote the C^*-algebra generated by a. By Diní's theorem, $\{a_+^{1/2+1/n}\}_n$ tends uniformly to $a_+^{1/2}$ in $\mathfrak{C}$. Hence, the norm limit

$u = \lim_n x_n \in \mathfrak{T}$ exists and satisfies $\mathfrak{a}(u, u) = \lim_n \mathfrak{a}(x_n, x_n) = a_+$. Similarly, $v = \lim_n x \cdot a_-^{1/n} \in \mathfrak{T}$ and $\mathfrak{a}(v, v) = -a_-$. From $a_+ a_- = 0$ we conclude that $\mathfrak{a}(u, v) = 0$. Finally, representing $\mathfrak{C} \oplus \mathbb{C}$ as the continuous functions on $[-1, 1]$, we easily obtain that $\{e_n\}_n = \{a_+^{1/n} + a_-^{1/n}\}_n$ is an approximate identity for $\mathfrak{C}$. This gives us

$$\mathfrak{a}(y, u + v) = \mathfrak{a}(y, \lim_n x \cdot e_n) = \lim_n \mathfrak{a}(y, x)\, e_n = \mathfrak{a}(y, x), \qquad y \in \mathfrak{T},$$

and hence $x = u + v$.

3.6. LEMMA. *For each $x \in \mathfrak{X}_\pm$ there exists $x' \in \mathfrak{X}_\pm$ satisfying $\mathfrak{a}(x', x') = \pm(\pm\mathfrak{a}(x, x))^{1/2}$ and $x = x' \cdot (\pm\mathfrak{a}(x, x))^{1/4}$.*

Proof. It suffices to consider the case $x \in \mathfrak{X}_+$. First, we have

LEMMA 3.6.1. *The ternary C^*-ring $\mathfrak{T}$ generated by x is contained in $\mathfrak{X}_+$, and $(\mathfrak{T}, \mathfrak{a} \mid \mathfrak{T} \times \mathfrak{T})$ is a Hilbert $\mathfrak{C}$-module, where $\mathfrak{C}$ denotes the C^*-algebra generated by $a = \mathfrak{a}(x, x)$.*

Proof. Indeed, define inductively $x_1 = x$, $x^{2i+1} = (x^{2i} - 1, x, x)$, $i \in \mathbb{N}$, and let $y = \sum_{i=0}^n \lambda_i x^{2i+1}$ be a ternary polynomial in x; then

$$\begin{aligned}\mathfrak{a}(y, y) &= \sum_{i,j} \lambda_i \bar{\lambda}_j \mathfrak{a}(x^{2i+1}, x^{2j+1}) = \sum_{i,j} \lambda_i \bar{\lambda}_j \mathfrak{a}(x \cdot a^i, x \cdot a^j) \\ &= \sum_{i,j} \lambda_i \bar{\lambda}_j a^j a a^i = \left(\sum_{i=0}^n \lambda_i a^i\right)\left(\sum_{i=0}^n \lambda_i a^i\right)^* a \geqslant 0\end{aligned}$$

by 3.2(2), hence, $\mathfrak{a}(y, y) \geqslant 0$ for all $y \in \mathfrak{T}$ so that 3.6.1 follows.

Applying Theorem 2.6 to the Hilbert $\mathfrak{C}$-module $\mathfrak{T}$ in 3.6.1, we find Hilbert spaces H and K, a linear isometry $U: \mathfrak{T} \to \mathscr{L}(H, K)$ and a faithful $*$-representation $\{\pi, H\}$ of $\mathfrak{C}$ such that

$$U(y \cdot a) = U(y)\, \pi(a), \qquad U(z)^* U(y) = \pi(\mathfrak{a}(y, z)), \qquad y, z \in \mathfrak{T}, \quad a \in \mathfrak{C}.$$

Now, let R be the partial isometry of the polar decomposition of $X = U(x)$. By Notations 1.1 and Proposition 1.2, $Y \mapsto R^* Y$ is an isomorphism of $U(\mathfrak{T})$ onto $\pi(\mathfrak{C})$. Defining $x' = U^{-1}(R\, |X|^{1/2})$ we obtain

$$\mathfrak{a}(x', x') = \pi^{-1}(U(x')^* U(x')) = \pi^{-1}(|X|) = \mathfrak{a}(x, x)^{1/2}$$

and

$$x' \cdot \mathfrak{a}(x, x)^{1/4} = U^{-1}(R\, |X|^{1/2} \pi(\mathfrak{a}(x, x))^{1/4}) = U^{-1}(X) = x.$$

3.7. PROPOSITION. *$\mathfrak{X}_+$ and $\mathfrak{X}_-$ are ternary C^*-subrings of $\mathfrak{X}$, and $\mathfrak{X} = \mathfrak{X}_+ \oplus \mathfrak{X}_-$. The associated projection $P: \mathfrak{X} \to \mathfrak{X}_+$ is self-adjoint relative to $\mathfrak{a}$ and $\mathfrak{b}$.*

Proof. We first show that

$$\mathfrak{X}_\pm = \{x \in \mathfrak{X} \mid \mathfrak{b}(x, x) \gtreqless 0\}. \tag{3.2}$$

So, let $x \in \mathfrak{X}_+$ and choose $x' \in \mathfrak{X}_+$ according to Lemma 3.6. Then

$$\begin{aligned}\mathfrak{b}(x', x')^2 &= \mathfrak{b}(\mathfrak{b}(x', x') \cdot x', x') = \mathfrak{b}(x' \cdot \mathfrak{a}(x', x'), x') \\ &= \mathfrak{b}(x' \cdot \mathfrak{a}(x, x)^{1/2}, x') \\ &= \mathfrak{b}(x' \cdot \mathfrak{a}(x, x)^{1/4}, x' \cdot \mathfrak{a}(x, x)^{1/4}) = \mathfrak{b}(x, x),\end{aligned}$$

and thus, $\mathfrak{b}(x, x) \geqslant 0$. Similarly, $\mathfrak{b}(x, x) \leqslant 0$ for each $x \in \mathfrak{X}_-$. Now, Eq. (3.2) follows from the identity $\mathfrak{a} = \mathfrak{c}$ in 3.4(3) and the fact that the positive cones of $\mathfrak{A}_\sim$ and $\mathfrak{A}$ coincide.

Let $x \in \mathfrak{X}_+$, $y \in \mathfrak{X}_-$. Then $y \cdot a \in \mathfrak{X}_-$, where $a = \mathfrak{a}(x, x)^{1/2}$. By 3.4(1), $0 \geqslant \mathfrak{b}(y \cdot a, y \cdot a) = \mathfrak{b}(y \cdot \mathfrak{a}(x, x), y) = \mathfrak{b}(\mathfrak{b}(y, x) \cdot x, y) = \mathfrak{b}(x, y)^* \mathfrak{b}(x, y)$ and hence, $\mathfrak{b}(x, y) = 0$. Therefore, $\mathfrak{b}(\mathfrak{X}_+, \mathfrak{X}_-) = \{0\}$.

Next, let $x, y \in \mathfrak{X}_+$. By Lemma 3.5, we have $x + y = u + v$ with $u \in \mathfrak{X}_+$, $v \in \mathfrak{X}_-$. Since $\mathfrak{a}(\mathfrak{X}_+, \mathfrak{X}_-) = \{0\}$ we obtain $0 = \mathfrak{a}(x, v) = \mathfrak{a}(y, v) = \mathfrak{a}(u, v)$. Hence, $\mathfrak{a}(v, v) = \mathfrak{a}(x + y - u, v) = 0$ and so, $x + y = u \in \mathfrak{X}_+$. Moreover, using $\mathfrak{a}(x \cdot a, x \cdot a) = a^* \mathfrak{a}(x, x) a \geqslant 0$, $x \in \mathfrak{X}_+$, $a \in \mathfrak{A}$, we conclude that

$$\mathfrak{a}((x, y, z), (x, y, z)) = \mathfrak{a}(x \cdot \mathfrak{a}(z, y), x \cdot \mathfrak{a}(z, y)) \geqslant 0, \qquad x, y, z \in \mathfrak{X}_+.$$

Thus, $\mathfrak{X}_+$ is a ternary subring of $\mathfrak{X}$ which is norm closed, because the positive cone of $\mathfrak{A}$ is closed and $\mathfrak{a}$ is continuous. Similarly, $\mathfrak{X}_-$ is a ternary C^*-subring of $\mathfrak{X}$. Finally, since $\mathfrak{X}_+$ and $\mathfrak{X}_-$ are $\mathfrak{a}$- and $\mathfrak{b}$-orthogonal the projection P is $\mathfrak{a}$-and $\mathfrak{b}$-self-adjoint.

Proof of 3.1. We start with 3.1(1). Define $\mathfrak{a}_+ = \mathfrak{a} \mid \mathfrak{X}_+ \times \mathfrak{X}_+$ and let $\mathfrak{A}_1$ denote the C^*-algebra generated by $\mathfrak{a}_+(\mathfrak{X}_+, \mathfrak{X}_+)$. By Proposition 3.7, $(\mathfrak{X}_+, \mathfrak{a}_+)$ is a faithful Hilbert $\mathfrak{A}_1$-module. In view of Theorem 2.6, there exist Hilbert spaces H_+ and K_+ and an isometry $U_+: \mathfrak{X}_+ \to \mathscr{L}(H_+, K_+)$ such that $\mathfrak{a}_+(x, y) \mapsto U_+(y)^* U_+(x)$ provides a faithful $*$-representation of $\mathfrak{A}_1$ on H_+ and $U_+((x, y, z)) = U_+(x \cdot \mathfrak{a}_+(z, y)) = U_+(x)\, U_+(y)^* U_+(z)$. Hence, $\mathfrak{X}_+$ is isomorphic to a TRO. By considering $-\mathfrak{a} \mid \mathfrak{X}_- \times \mathfrak{X}_-$ we see that $\mathfrak{X}_-$ is anti-isomorphic to a TRO.

To show uniqueness, assume $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are ternary C^*-subrings of $\mathfrak{X}$ with $\mathfrak{X} = \mathfrak{X}_1 \oplus \mathfrak{X}_2$ such that representations $U_i: \mathfrak{X}_i \to \mathscr{L}(H_i, K_i)$ $i = 1, 2$, of the prescribed kind exist. Put $\mathfrak{A}_1 = [U_1(\mathfrak{X}_1)]_{H_1}$. Then $\mathfrak{X}_1$ becomes a right Banach $\mathfrak{A}_1$-module by

$$x \cdot A = U_1^{-1}(U_1(x)A), \qquad x \in \mathfrak{X}_1, \quad A \in \mathfrak{A}_1.$$

Define a continuous Hermitian form $\mathfrak{a}_1 : \mathfrak{X}_1 \times \mathfrak{X}_1 \to \mathfrak{A}_1$ according to

$$\mathfrak{a}_1(x, y) = U_1(y)^* U_1(x), \qquad x, y \in \mathfrak{X}_1.$$

Since $x \cdot \mathfrak{a}_1(z, y) = U_1^{-1}(U_1(x)\, U_1(y)^* U_1(z)) = (x, y, z)$, the uniqueness argument of Proposition 3.2 implies that $\mathfrak{a}(x, y) \mapsto \mathfrak{a}_1(x, y)$ defines a $*$-isomorphism of the C^*-algebra generated by $\mathfrak{a}(\mathfrak{X}_1, \mathfrak{X}_1)$ onto $\mathfrak{A}_1$. Because of $\mathfrak{a}_1(x, x) \geqslant 0$ we obtain $\mathfrak{X}_1 \subset \mathfrak{X}_+$. Similarly, $\mathfrak{X}_2 \subset \mathfrak{X}_-$. Thus, $\mathfrak{X}_+ = \mathfrak{X}_1$ and $\mathfrak{X}_- = \mathfrak{X}_2$.

To establish 3.1(2), put $T = 2P - I$. Since, by 3.7, T is $\mathfrak{a}$- (respectively, $\mathfrak{b}$-) self-adjoint it is easy to check that T has the required properties. Moreover, $\mathfrak{a}_T : (x, y) \mapsto \mathfrak{a}(Tx, y)$ is an $\mathfrak{A}$-valued inner product on $\mathfrak{X}$, because

$$\mathfrak{a}_T(x, x) = \mathfrak{a}(2Px - x, x) = 2\mathfrak{a}(x, x)_+ - \mathfrak{a}(x, x) = |\mathfrak{a}(x, x)| \geqslant 0, \qquad x \in \mathfrak{X}.$$

Using Eq. (2.6) with $(\mathfrak{H}, \langle\cdot \mid \cdot\rangle) = (\mathfrak{X}, \mathfrak{a}_T)$, we get Hilbert spaces H and K and an isometry $U : \mathfrak{X} \to \mathscr{L}(H, K)$ such that $U(x)\, U(y)^* U(z) = U(x \cdot \mathfrak{a}_T(z, y)) = U((x, y), Tz)) = U \circ T((x, y, z))$.

Finally, suppose that $S : \mathfrak{X} \to \mathfrak{X}$ and $V : \mathfrak{X} \to \mathscr{L}(H_S, K_S)$ have the same properties as T and U. Obviously, $S \in \mathfrak{A}_{\mathfrak{a}}(\mathfrak{X})$ is self-adjoint. Hence, $S = E - F$, $1 = E + F$, $EF = 0$ with self-adjoint idempotents $E, F \in \mathfrak{A}_{\mathfrak{a}}(\mathfrak{X})$. In particular, E and F are $\mathfrak{A}$-module homomorphisms. Thus, $\mathfrak{X}_1 = E(\mathfrak{X})$ and $\mathfrak{X}_2 = F(\mathfrak{X})$ are ternary C^*-subrings of $\mathfrak{X}$, and $\mathfrak{X} = \mathfrak{X}_1 \oplus \mathfrak{X}_2$. Furthermore, $V_1 = V | \mathfrak{X}_1$ is an isomorphism. Because of $V((x, y, z)) = -V \circ S(F(x, y, z)) = -V(x)\, V(y)^* V(z)$ for all $x, y, z \in \mathfrak{X}_2$, $V_2 = V | \mathfrak{X}_2$ is an anti-isomorphism. Now, 3.1(1) implies $\mathfrak{X}_+ = \mathfrak{X}_1$, $\mathfrak{X}_- = \mathfrak{X}_2$ and $S = P - (I - P) = T$.

To give examples, first let $\mathfrak{R}$ be a norm closed TRO between H and K. Then $(\mathfrak{R}, (\cdot, \cdot, \cdot), \|\cdot\|)$ is a ternary C^*-ring, where $\|\cdot\|$ denotes the operator norm and $(\cdot, \cdot, \cdot)$ is given by $(A, B, C) = AB^*C$. In particular, this way each C^*-algebra is a ternary C^*-ring. If $(\mathfrak{H}, \langle\cdot \mid \cdot\rangle)$ is a Hilbert $\mathfrak{A}$-module (or, more specifically, a complex Hilbert space), then $(\mathfrak{H}, (\cdot, \cdot, \cdot), \|\cdot\|_{\mathfrak{H}})$ is a ternary C^*-ring with $(x, y, z) = x \cdot \langle z \mid y\rangle$. In all these situations we obtain $T = I$ if theorem 3.1 is applied. It is, however, easy to see that T and I can be different.

3.8. Example. Suppose Ω is a compact Hausdorff space which contains a nonvoid open and closed set $\Omega' \neq \Omega$. Define $\chi : \Omega \to \{0, 1\}$ by $\chi(t) = 1$ iff $t \in \Omega'$, and put $(f, g, h) = f\bar{g}h(2\chi - 1)$, $f, g, h \in \mathscr{C}(\Omega)$. Then $(\mathscr{C}(\Omega), (\cdot, \cdot, \cdot), \|\cdot\|_{\sup})$ is a ternary C^*-ring, whose decomposition in the sense of 3.1(1) is given by $\mathscr{C}(\Omega)_+ = \mathscr{C}(\Omega)\chi = \mathscr{C}(\Omega')$, $\mathscr{C}(\Omega)_- = \mathscr{C}(\Omega\backslash\Omega')$. Consequently, $Tf = f(2\chi - 1)$ for each $f \in \mathscr{C}(\Omega)$ and so $T \neq 1$.

The problem of characterizing the ternary C^*-rings which yield $T = 1$ remains open in the following. Let us mention two further points. Since, in view of Proposition 3.4, $\mathfrak{A}_{\mathfrak{b}}(\mathfrak{X})$ commutes with $\mathfrak{A}_{\mathfrak{a}}(\mathfrak{X})$ and hence, $T = 2P - 1$ belongs to the center of $M(\mathfrak{A})$ by Proposition 3.7, we see that $T \in \{1, -1\}$ if $\mathfrak{A}$ is a factor. If $\mathfrak{X}$ contains a "right unit," i.e., an element r such that $x = (x, r, r)$, $x \in \mathfrak{X}$, then $T = 1$. Indeed, $x = x \cdot \mathfrak{a}(r, r)$ implies that $\mathfrak{a}(r, r)$ is a unit in $\mathfrak{A}$. Therefore, $r \in \mathfrak{X}_+$. Since $\mathfrak{b}(\mathfrak{X}_-, \mathfrak{X}_+) = \{0\}$ (cf. the proof of 3.7) we have that $x = (x, r, r) = \mathfrak{b}(x, r) \cdot r = 0$ for all $x \in \mathfrak{X}_-$.

4. The Representation Theorem for Ternary W^*-Rings

A ternary W^*-ring $(\mathfrak{X}, (\cdot,\cdot,\cdot), \|\cdot\|)$ with a predual $\mathfrak{X}_*$ and the weak* topology $\sigma = \sigma(\mathfrak{X}, \mathfrak{X}_*)$ will be called *normally isomorphic* (respectively, *normally anti-isomorphic*) *to a* TRO, if there are Hilbert spaces H and K and a linear isometry $U: \mathfrak{X} \to \mathscr{L}(H, K)$ which is a homeomorphism relative to σ and the weak-operator topology and which satisfies

$$U((x, y, z)) = U(x)\, U(y)^* U(z), \qquad x, y, z \in \mathfrak{X}$$

(respectively, $U((x, y, z)) = -U(x)\, U(y)^* U(z)$, $x, y, z \in \mathfrak{X}$). Because of the weak* compactness of $D_1(\mathfrak{X})$ and the density theorem 1.4, $U(\mathfrak{X})$ is a weakly closed TRO.

The main result of this section consists now in the following:

4.1. Representation Theorem. *Let* $(\mathfrak{X}, (\cdot,\cdot,\cdot), \|\cdot\|)$ *be a ternary* W^*-*ring*, $\mathfrak{X} = \mathfrak{X}_+ \oplus \mathfrak{X}_-$ *its decomposition according to* 3.1(1), *and* T *the operator of* 3.1(2). *Then*:

(1) $\mathfrak{X}_+$ *(respectively,* $\mathfrak{X}_-$*) is* σ*-closed and a ternary* W^**-subring of* $\mathfrak{X}$ *which is normally isomorphic (respectively, normally anti-isomorphic) to a* TRO;

(2) T *is* σ*-continuous, and the ternary* W^**-ring* $(\mathfrak{X}, T \circ (\cdot,\cdot,\cdot), \|\cdot\|)$ *is normally isomorphic to a* TRO.

In the proof we use the C^*-algebra $\mathfrak{A}$ and the $\mathfrak{A}$-form $\mathfrak{a}$ provided by Proposition 3.2. The essential step is Proposition 4.9 which states that the double centralizer algebra $M(\mathfrak{A})$ of $\mathfrak{A}$ is a W^*-algebra and that $\mathfrak{X}$, endowed with the positive form $\mathfrak{a}_T$: $(x, y) \mapsto \mathfrak{a}(Tx, y)$, is a self-dual Hilbert $M(\mathfrak{A})$-module. Theorem 4.1 is then a consequence of Theorem 2.8. Subsequently, we obtain that $M(\mathfrak{A})_*$ is equal to a quotient of $\mathfrak{X} \hat{\otimes}_\gamma \mathfrak{X}_*$ and that $\mathfrak{X}_*$

coincides with $\mathfrak{X}_c \otimes M(\mathfrak{A})_*$. In particular, $\mathfrak{X}_*$ is uniquely determined up to isomorphism.

To establish Proposition 4.9 we proceed as follows: Replacing $\mathfrak{a}$ by $\mathfrak{a}_T$ we suppose that $(\mathfrak{X}, \mathfrak{a})$ is a Hilbert $\mathfrak{A}$-module. Let $(\mathfrak{X}'', \mathfrak{a})$ denote its self-dual completion and σ'' the weak toplogy of $\mathfrak{X}''$ (cf. Proposition 2.2 and Definition 2.3). Applying Theorem 2.6 we regard $\mathfrak{X}$ as a norm closed TRO between H and K with weak closure $\mathfrak{X}''$ such that $[\mathfrak{X}]_H$ (respectively, $\{\mathfrak{X}\}_H$) is equal to $\mathfrak{A}$ (respectively, $\mathfrak{A}^{**}$) and σ'' and $\sigma(\mathfrak{A}^{**}, \mathfrak{A}^*)$ are just the weak-operator topologies. Moreover, it is easy to see that $[\mathfrak{X}]_K$ coincides with the C^*-algebra $\mathfrak{B}$ generated by $\mathfrak{b}(\mathfrak{X}, \mathfrak{X})$ (cf. Proposition 3.4). Our aim is now to construct two weak* continuous projections $\Phi: \mathfrak{X}'' \to \mathfrak{X}$ and $\Psi: \mathfrak{A}^{**} \to M(\mathfrak{A})$ of norm one which, for a central projection $e \in \mathfrak{A}^{**}$ satisfy

$$\operatorname{kernel}(\Phi) = \mathfrak{X}''(1-e) \qquad \text{and} \qquad \operatorname{kernel}(\Psi) = \mathfrak{A}^{**}(1-e).$$

Here, we use the fact that $M(\mathfrak{A})$ is a C^*-subalgebra of $\mathfrak{A}^{**}$. The idea of proof in the following is taken from [9, Lemma 1.7.6]:

4.2. Lemma. *If $E \in [\mathfrak{X}]_H$ and $E' \in [\mathfrak{X}]_K$ are self-adjoint projections, then $\mathfrak{X} \cdot E$, $E' \cdot \mathfrak{X}$ are $(1-E') \cdot \mathfrak{X}$ are σ-closed in $\mathfrak{X}$.*

Proof. Let $\{x_\alpha\}_\alpha \subset D_1(\mathfrak{X}) \cap \mathfrak{X} \cdot E$ be a net which tends to $x \in \mathfrak{X}$ in σ. We have to show that $x = x \cdot E$.

Suppose by way of contradiction that $x' = x \cdot (1-E) \neq 0$. In view of Proposition 1.2, the partial isometry r of the polar decomposition of x' belongs to $\mathfrak{X}''$. According to Proposition 1.4, there is a net $\{y_\beta\}_\beta \subset D_1(\mathfrak{X})$ which tends weakly to r. Since $\tau \mapsto \tau \cdot (1-E)$ is weakly continuous on $\mathfrak{X}''$, we may assume that $y_\beta = y_\beta \cdot (1-E)$ for all β. By Notations 1.1 and Proposition 1.2, x' is positive in the W^*-algebra $\mathfrak{X}''_{(r)}$. Hence, there exists $\lambda > 0$ satisfying

$$\lambda + n \leqslant \|x' + nr\| = \liminf_\beta \|x' + ny_\beta\|, \qquad n \in \mathbb{N}.$$

On the other hand, for each $y \in D_1(\mathfrak{X})$ with $y = y \cdot (1-E)$ we have

$$\|x_\alpha + ny\| = \|\mathfrak{b}(x_\alpha, x_\alpha) + n^2 \mathfrak{b}(y, y)\|^{1/2} \leqslant (1 + n^2)^{1/2}, \qquad n \in \mathbb{N}, \quad \text{all} \quad \alpha,$$

and thus, together with $x + ny_\beta = \sigma - \lim_\alpha (x_\alpha + ny_\beta)$,

$$\|x' + ny_\beta\| = \|(x + ny_\beta) \cdot (1-E)\| \leqslant (1+n^2)^{1/2}, \quad n \in \mathbb{N}, \quad \text{all} \quad \beta,$$

which leads to $\lambda + n \leqslant (1+n^2)^{1/2}$, $n \in \mathbb{N}$, a contradiction.

The proof in the case of E' and $1 - E'$ is the same.

4.3. Proposition. *Let $f \in \mathfrak{X}_*$. Then there exist a partial isometry $r \in \mathfrak{X}$ and positive functionals $\varphi \in \mathfrak{A}^*$, $\psi \in \mathfrak{B}^*$ such that*

(1) $\varphi(\mathfrak{a}(x, r)) = f(x) = \psi(\mathfrak{b}(x, r)), x \in \mathfrak{X}$,

(2) $f(r \cdot a) = \varphi(a), a \in \mathfrak{A}$,

(3) $f(b \cdot r) = \psi(b), b \in \mathfrak{B}$.

Proof. We can suppose that $\|f\| = 1$. Since $D_1(\mathfrak{X})$ is compact in σ, there is an extreme point r of $D_1(\mathfrak{X})$ with $f(r) = 1$. In view of Lemma 1.3, r is a partial isometry. Now define $\varphi \in \mathfrak{A}^*$ by $\varphi(a) = f(r \cdot a)$. From $\varphi(\mathfrak{a}(r, r)) = f(r \cdot \mathfrak{a}(r, r)) = f(r) = 1$ we conclude that $\|\varphi\| = 1$. Hence, due to [9, Proposition 1.5.2], φ is positive. Moreover, $\varphi(\mathfrak{a}(x, r)) = f(r \cdot \mathfrak{a}(x, r)) = f(\mathfrak{b}(r, r) \cdot x)$, $x \in \mathfrak{X}$. To see that $f(x) = f(\mathfrak{b}(r, r)) \cdot x)$, we argue by contradiction. Put $E' = \mathfrak{b}(r, r)$ and suppose that $f(x) \neq 0$ for some $x \in (1 - E') \cdot \mathfrak{X}$. By Lemma 4.2, $(1 - E') \cdot \mathfrak{X}$ is σ-closed and hence, a ternary W^*-ring. As above, there is a partial isometry $r' \in (1 - E') \cdot \mathfrak{X}$ with $f(r') > 0$. Thus,

$$\begin{aligned}(1 + n^2)^{1/2} &\geqslant \| n^2\mathfrak{a}(r, r) + \mathfrak{a}(r', r')\|^{1/2} = \|\mathfrak{a}(nr + r', nr + r')\|^{1/2} \\ &= \|nr + r'\| \geqslant |f(nr + r')| = n + f(r'), \qquad n \in \mathbb{N},\end{aligned}$$

a contradiction. The functional ψ is found similarly.

4.4. Lemma. *Each $f \in \mathfrak{X}_*$ has a unique extension to a weakly continuous functional f'' on $\mathfrak{X}''$. The adjoint $\Phi\colon \mathfrak{X}'' \to \mathfrak{X}$ of $f \mapsto f''$ is a linear, σ''-σ-continuous projection of norm one.*

Proof. Let $f \in \mathfrak{X}_*$ and write $f = \varphi(\mathfrak{a}(\cdot, r))$ is accordance with 4.3(1), (2). Define f'' on $\mathfrak{X}''$ by $f''(\tau) = \varphi(\mathfrak{a}(\tau, r))$. Then f'' is a weakly continuous functional on $\mathfrak{X}''$, because $\mathfrak{a}(\tau, r) = \tau(r)^*$ for all $\tau \in \mathfrak{X}''$ by 2.2(4). Obviously, f'' is an extension of f. Since, because of Proposition 1.4, $D_1(\mathfrak{X})$ is weakly dense in $D_1(\mathfrak{X}'')$ the extension is unique and satisfies $\|f\| = \|f''\|$.

We are now going to derive the module properties of Φ and to show that the kernel of Φ is a two-sided ideal in the sense of Lemma 1.5. First, we look at the behaviour of Φ on C^*-algebras which are associated with partial isometries in $\mathfrak{X}$.

4.5. Lemma. *Let $r \in \mathfrak{X}$ be a partial isometry and $\mathfrak{X}_{(r)}$ (respectively, $\mathfrak{X}''_{(r)}$) the C^*-algebra (respectively, W^*-algebra) associated with r in $\mathfrak{X}$ (respectively, $\mathfrak{X}''$) according to Notations 1.1. Then, $\Phi(\mathfrak{X}''_{(r)}) = \mathfrak{X}_{(r)}$. In particular, $\mathfrak{X}_{(r)}$ is a W^*-algebra with predual $\{f | \mathfrak{X}_{(r)} | f \in \mathfrak{X}_*\}$.*

Proof. We start with the following claim:

Claim 4.5.1. Let $x \in \mathfrak{X}$. If $f(x) \geqslant 0$ for all $f \in \mathfrak{X}^*$ whose restriction to $\mathfrak{X}_{(r)}$ is a positive functional, then x is positive in $\mathfrak{X}_{(r)}$.

By the Hahn–Banach theorem, the state space of $\mathfrak{X}_{(r)}$ is equal to $\{f|\mathfrak{X}_{(r)} | f \in \mathfrak{X}^*, f(r) = 1 = \|f\|\}$. Hence, it suffices to show that $x \in \mathfrak{X}_{(r)}$. Suppose that $x' = (1 - E') \cdot x \neq 0$, where E' denotes the range projection of r. Then there is $f \in \mathfrak{X}^*$ with $f(x') = -1$. Since the functional $g\colon y \mapsto f(y - E' \cdot y)$ vanishes on $\mathfrak{X}_{(r)}$ we obtain $g(x) \geqslant 0$ which contradicts $g(x) = f(x') = -1$. Similarly, $x = x \cdot \mathfrak{a}(r, r)$.

It follows that $\Phi(\mathfrak{X}''_{(r)}) \subset \mathfrak{X}_{(r)}$. Indeed, let $\tau \in \mathfrak{X}''_{(r)}$ be positive and fix $f \in \mathfrak{X}^*$ satisfying $f(r) = 1 = \|f\|$. If g denotes $f \circ \Phi$ restricted to $\mathfrak{X}''_{(r)}$, then $g(r) = f(r) = 1$ and $\|g\| \leqslant 1$. Hence, g is a state. Therefore, $g(\tau) \geqslant 0$ and so, $f(\Phi\tau) \geqslant 0$. By Claim 4.5.1, $\Phi(\tau) \in \mathfrak{X}_{(r)}$.

Observe next that $\mathfrak{X}''_{(r)}$ is weakly closed in $\mathfrak{X}''$. Since $D_1(\mathfrak{X}_{(r)}) = \Phi(D_1(\mathfrak{X}''_{(r)}))$ this ball is compact in σ and thus, $\mathfrak{X}_{(r)}$ is σ-closed.

4.6. Lemma. *For all $A \in M(\mathfrak{A})$ and $b \in \mathfrak{B}$, the mappings $x \mapsto x \cdot A$ and $x \mapsto b \cdot x$ are σ-continuous on $\mathfrak{X}$.*

Proof. It suffices to show that $x \mapsto x \cdot U$ is σ-continuous for each unitary $U \in M(\mathfrak{A})$. So, let $\{x_\alpha\}_\alpha \subset \mathfrak{X}$ be a net tending in σ to zero and fix $f \in \mathfrak{X}_*$. We must prove that $\lim_\alpha f(x_\alpha \cdot U) = 0$.

Write $f = \psi(\mathfrak{b}(\cdot, r))$ according to Proposition 4.3. Put $E' = \mathfrak{b}(r, r)$ and $r' = r \cdot U^*$. Then r' is a partial isometry with range projection E'. Define $\mathfrak{C} = E'\mathfrak{B}E'$. In view of Proposition 1.2, a *-isomorphism of $\mathfrak{C}$ onto $\mathfrak{X}_{(r)}$ (respectively, $\mathfrak{X}_{(r')}$) is given by $b \mapsto b \cdot r$ (respectively, $b \mapsto b \cdot r'$). Using Lemma 4.5 we obtain that $\mathfrak{C}$ is a W^*-algebra with predual $\{b \mapsto g(b \cdot r) \mid g \in \mathfrak{X}_*\} = \{b \mapsto g(b \cdot r') \mid g \in \mathfrak{X}_*\}$. Hence, ψ is normal on $\mathfrak{C}$ and so, $\psi(b) = h(b \cdot r')$, $b \in \mathfrak{C}$ for some $h \in \mathfrak{X}_*$. By Lemma 4.2, the mappings $x \mapsto E' \cdot x$ and $x \mapsto x \cdot \mathfrak{a}(r', r')$ are σ-continuous on $\mathfrak{X}$. Thus,

$$\lim_\alpha f(x_\alpha \cdot U) = \lim_\alpha \psi(\mathfrak{b}(x_\alpha \cdot U, r)) = \lim_\alpha \psi(\mathfrak{b}(E' \cdot x_\alpha, r'))$$

$$= \lim_\alpha h(\mathfrak{b}(E' \cdot x_\alpha, r') \cdot r') = \lim_\alpha h(E' \cdot x_\alpha \cdot \mathfrak{a}(r', r')) = 0.$$

4.7. Corollary. *Let Φ be the projection defined in Lemma* 4.4. *Then*

(1) *$\Phi(\tau \cdot A) = \Phi(\tau) \cdot A$ and $\Phi(b \cdot \tau) = b \cdot \Phi(\tau)$ for all $\tau \in \mathfrak{X}''$, $A \in M(\mathfrak{A})$ and $b \in \mathfrak{B}$.*

(2) *The kernel of Φ,* $\ker(\Phi)$, *is of the form* $\ker(\Phi) = \mathfrak{X}'' \cdot (1 - e)$, *where e is a (uniquely determined) central projection in $\mathfrak{A}^{**}$.*

Proof. Let $\tau \in \mathfrak{X}''$. Then τ is the weak limit of a net $\{x_\alpha\}_\alpha \subset \mathfrak{X}$, and

$\{x_\alpha \cdot A\}_\alpha$ tends weakly to $\tau \cdot A$ for each $A \in M(\mathfrak{A})$. Using $\Phi(\tau) = \sigma - \lim_\alpha x_\alpha$ and Lemma 4.6 we obtain

$$\Phi(\tau \cdot A) = \sigma - \lim_\alpha \Phi(x_\alpha \cdot A) = \sigma - \lim_\alpha x_\alpha \cdot A = \Phi(\tau) \cdot A, \qquad A \in M(\mathfrak{A}).$$

The second statement in (1) follows similarly. Now, (1) implies that $\ker(\Phi) \cdot \mathfrak{A} \cup \mathfrak{B} \cdot \ker(\Phi) \subset \ker(\Phi)$. By 1.5, $\ker(\Phi) = \mathfrak{X}'' \cdot (I - e)$ with a unique central projection $e \in \{\mathfrak{X}\}_H = \mathfrak{A}^{**}$.

4.8. Lemma. *$M(\mathfrak{A})$ is a W^*-algebra. In particular, there exists a linear weak* continuous projection $\Psi\colon \mathfrak{A}^{**} \to M(\mathfrak{A})$ such that*

(1) $\Psi(\mathfrak{a}(\tau, x)) = \mathfrak{a}(\Phi(\tau), x)$, $\tau \in \mathfrak{X}''$, $x \in \mathfrak{X}$;

(2) $\ker(\Psi) = \mathfrak{A}^{**}(I - e)$, *where e is the projection in* 4.7(2).

Proof. Let $A \in \mathfrak{A}^{**}$. By 4.7(1), $\Phi_A : x \mapsto \Phi(x \cdot A)$ is a $\mathfrak{B}$-module homomorphism on $\mathfrak{X}$. We are going to show that $\Phi_A \in \mathfrak{A}_{\mathfrak{b}}(\mathfrak{X})$ and then define $\Psi(A) = \Phi_A$.

To this end we write Ω for the set of all bounded $\mathfrak{B}$-module homomorphisms on $\mathfrak{X}$. Observe that $M(\mathfrak{A}) \subset \Omega$ by 3.4(4). If Γ denotes the weak topology on Ω generated by all functionals $S \mapsto f(Sx)$, $f \in \mathfrak{X}_*$, $x \in \mathfrak{X}$, then we have:

Lemma 4.8.1. *$\mathfrak{a}$ is separately $(\sigma \times \sigma)$-Γ-continuous.*

Proof. If $\{x_\alpha\}_\alpha \subset \mathfrak{X}$ is a net tending to $x \in \mathfrak{X}$ in σ, then Lemma 4.6 shows

$$\lim_\alpha f(y \cdot \mathfrak{a}(x_\alpha, z)) = \lim_\alpha f(\mathfrak{b}(y, z) \cdot x_\alpha) = f(\mathfrak{b}(y, z) \cdot x)$$
$$= f(y \cdot \mathfrak{a}(x, z)), \qquad f \in \mathfrak{X}_*, \quad y, z \in \mathfrak{X},$$

and thus, $\mathfrak{a}(x, z) = \Gamma\text{-}\lim_\alpha \mathfrak{a}(x_\alpha, z)$. On the other hand, with $f = \varphi(\mathfrak{a}(\cdot, r))$ according to 4.3(1), (2) we conclude that

$$\begin{aligned} f(y \cdot \mathfrak{a}(z, x_\alpha)) &= \varphi(\mathfrak{a}(y, r)\, \mathfrak{a}(z, x_\alpha)) = \overline{\varphi(\mathfrak{a}(x_\alpha, z)\, \mathfrak{a}(r, y))} \\ &= \overline{\varphi(\mathfrak{a}(r, r)\, \mathfrak{a}(x_\alpha, z)\, \mathfrak{a}(r, y))} = \overline{\varphi(\mathfrak{a}(r \cdot \mathfrak{a}(x_\alpha, z)\, \mathfrak{a}(r, y), r))} \\ &= \overline{\varphi(\mathfrak{a}(\mathfrak{b}(r, z) \cdot x_\alpha \cdot \mathfrak{a}(r, y), r))} = \overline{f(\mathfrak{b}(r, z) \cdot x_\alpha \cdot \mathfrak{a}(r, y))} \\ &\xrightarrow[\alpha]{} \overline{f(\mathfrak{b}(r, z) \cdot x \cdot \mathfrak{a}(r, y))} = \cdots = f(y \cdot \mathfrak{a}(z, x)). \end{aligned}$$

Hence, $\mathfrak{a}(z, x) = \Gamma\text{-}\lim_\alpha \mathfrak{a}(z, x_\alpha)$, $z \in \mathfrak{X}$, proving Lemma 4.8.1.

Now, let A be the weak*-limit of a net $\{a_\alpha\}_\alpha \subset \mathfrak{A}$. Then

$$\mathfrak{a}(z, \Phi_{A^*}(y)) = \Gamma\text{-}\lim_\alpha \mathfrak{a}(z, y \cdot a_\alpha^*), \qquad y, z \in \mathfrak{X},$$

which follows from Lemma 4.8.1, because $y \cdot A^* = \sigma''\text{-}\lim_\alpha y \cdot a_\alpha^*$ and so, $\Phi_{A^*}(y) = \sigma\text{-}\lim_\alpha y \cdot a_\alpha^*$. Therefore, we obtain

$$\begin{aligned}\mathfrak{b}(\Phi_A(x), y) \cdot z &= \Phi(x \cdot A) \cdot \mathfrak{a}(z, y) = \Phi(x \cdot A\mathfrak{a}(z, y)) \\ &= \sigma\text{-}\lim_\alpha x \cdot a_\alpha \mathfrak{a}(z, y) = \sigma\text{-}\lim_\alpha x \cdot \mathfrak{a}(z, y \cdot a_\alpha^*) \\ &= x \cdot \mathfrak{a}(z, \Phi_{A^*}(y)) = \mathfrak{b}(x, \Phi_{A^*}(y)) \cdot z,\end{aligned}$$

and consequently, $\Phi_A \in \mathfrak{A}_\mathfrak{b}(\mathfrak{X})$ and $(\Phi_A)^* = \Phi_{A^*}$.

Obviously, $\Psi: A \to \Phi_A$ is a *-anti-homomorphism of $\mathfrak{A}^{**}$ onto $\mathfrak{A}_\mathfrak{b}(\mathfrak{X}) = M(\mathfrak{A})_\sim$. Because of 4.7(1), Ψ is a projection. Moreover,

$$\ker(\Psi) = \{A \in \mathfrak{A}^{**} \mid \mathfrak{X} \cdot A \subset \ker(\Phi)\} = \mathfrak{A}^{**}(I - e).$$

Therefore, $M(\mathfrak{A})$ is a W^*-algebra which is isomorphic to $\mathfrak{A}^{**}e$, and Ψ is weak* continuous. Finally, suppose that $\tau = \sigma''\text{-}\lim_\alpha x_\alpha$ for a net $\{x_\alpha\}_\alpha \subset \mathfrak{X}$. Then (1) follows from

$$\begin{aligned}y \cdot \Psi(\mathfrak{a}(\tau, x)) &= \sigma\text{-}\lim_\alpha \Phi(y \cdot \mathfrak{a}(x_\alpha, x)) = \sigma\text{-}\lim_\alpha \mathfrak{b}(y, x) \cdot x_\alpha \\ &= \mathfrak{b}(y, x) \cdot \Phi(\tau) = y \cdot \mathfrak{a}(\Phi(\tau), x).\end{aligned}$$

4.9. Proposition. *Let $(\mathfrak{X}(\cdot,\cdot,\cdot), \|\cdot\|)$ be a ternary W^*-ring and $T: \mathfrak{X} \to \mathfrak{X}$ the operator belonging to it by* 3.1(2). *Then $M(\mathfrak{A})$ is a W^*-algebra, and $(\mathfrak{X}, \mathfrak{a}_T)$ is a self-dual Hilbert $M(\mathfrak{A})$-module, $\mathfrak{a}_T$ being defined by $\mathfrak{a}_T(x, y) = \mathfrak{a}(Tx, y)$.*

Proof. Let $\tau: \mathfrak{X} \to M(\mathfrak{A})$ be a bounded $M(\mathfrak{A})$-module homomorphism. Then $\tau_1 = \tau \circ \Phi$ is in $\mathfrak{X}''$, and 4.8(1) shows that

$$\tau(x) = \tau_1(x) = \Psi(\tau_1(x)^*)^* = \Psi(\mathfrak{a}_T(\tau_1, x))^* = \mathfrak{a}_T(x, \Phi(\tau_1)), \qquad x \in \mathfrak{X}.$$

Hence, $(\mathfrak{X}, \mathfrak{a}_T)$ is self-dual over $M(\mathfrak{A})$.

Proof of 4.1. By virtue of Proposition 3.7, the projection $P: \mathfrak{X} \to \mathfrak{X}_+$ belongs to the center of $M(\mathfrak{A})$. By Lemma 4.6, P is σ-continuous. Therefore, $\mathfrak{X}_+$ and $\mathfrak{X}_-$ are σ-closed, and $T = 2P - I$ is σ-continuous. Because of Proposition 4.9, $(\mathfrak{X}_\pm, \mathfrak{a} \mid \mathfrak{X}_\pm \times \mathfrak{X}_\pm)$ and $(\mathfrak{X}, \mathfrak{a}_T)$ are self-dual Hilbert modules over the W^*-algebra $M(\mathfrak{A})$. From Theorem 2.8 we obtain the desired isomorphisms.

4.10. Corollary. *Let $(\mathfrak{X}, (\cdot,\cdot,\cdot), \|\cdot\|)$ be a ternary W^*-ring and $\mathfrak{A}$, $\mathfrak{B}$ the C^*-algebras as above. Then $M(\mathfrak{A})_\sim$ is the set of all bounded $M(\mathfrak{B})$-module homomorphisms on $\mathfrak{X}$. In particular, $M(\mathfrak{A})$ is the dual space of the quotient $\mathfrak{X} \otimes_{M(\mathfrak{B})} \mathfrak{X}_*$ of $\mathfrak{X} \hat{\otimes}_\gamma \mathfrak{X}_*$.*

Proof. Define $\mathfrak{b}_T: \mathfrak{X} \times \mathfrak{X} \to M(\mathfrak{B})$ by $\mathfrak{b}_T(x, y) = \mathfrak{b}(Tx, y)$. If we apply the same line of reasoning by which Proposition 4.9 has been established to $(\mathfrak{X}, \mathfrak{B}, \mathfrak{b}_T)$ instead of $(\mathfrak{X}, \mathfrak{A}, \mathfrak{a}_T)$ we obtain that $(\mathfrak{X}, \mathfrak{b}_T)$ is a self-dual (left) Hilbert module over the W^*-algebra $M(\mathfrak{B})$. In view of [5, Corollary 3.5], the set of all bounded $M(\mathfrak{B})$-module homomorphisms is equal to $\mathfrak{A}_{\mathfrak{b}_T}(\mathfrak{X})$. Now, $\mathfrak{A}_{\mathfrak{b}_T}(\mathfrak{X}) = \mathfrak{A}_{\mathfrak{b}}(\mathfrak{X})$ and, by 3.4(3), $\mathfrak{A}_{\mathfrak{b}}(\mathfrak{X}) = M(\mathfrak{A}_{\sim}) = M(\mathfrak{A})_{\sim}$, and the first assertion follows. For the proof of the second statement, we observe that $\mathfrak{X}_*$ is a left Banach $M(\mathfrak{B})$-module. Hence, [7, Corollary 2.13] shows that $(\mathfrak{X} \otimes_{M(\mathfrak{B})} \mathfrak{X}_*)^* = M(\mathfrak{A})$.

4.11. Corollary. *Let $(\mathfrak{X}, (\cdot, \cdot, \cdot), \|\cdot\|)$ be a ternary W^*-ring. Then the predual of $\mathfrak{X}$ consists precisely of all functionals $x \mapsto \varphi(\mathfrak{a}_T(x, y))$, $y \in \mathfrak{X}$, $\varphi \in M(\mathfrak{A})_*$ and is therefore uniquely determined up to isomorphism.*

Proof. Let f be a weak* continuous functional on $\mathfrak{X}$. By Proposition 4.3, there exist $r \in \mathfrak{X}$ and $\varphi \in M(\mathfrak{A})^*$ such that $f = \varphi(\mathfrak{a}_T(\cdot, r))$ and $\varphi = f(r\, \cdot)$. In particular, φ is Γ-continuous on $M(\mathfrak{A})$, where Γ denotes the topology defined in the proof of Lemma 4.8. Since the projection Ψ: $\mathfrak{A}^{**} \to M(\mathfrak{A})$ is weak*-Γ-continuous we conclude that $\varphi \circ \Psi \in \mathfrak{A}^*$. Consequently, $\varphi \in M(\mathfrak{A})_*$ so that f has the desired representation.

On the other hand, fix $y \in \mathfrak{X}$, $\varphi \in M(\mathfrak{A})_*$ and put $g = \varphi(\mathfrak{a}_T(\cdot, y))$. Then Corollary 4.10 shows the existence of sequences $\{x_n\}_n \subset \mathfrak{X}$, $\{f_n\}_n \subset \mathfrak{X}_*$ satisfying $\sum_n \|x_n\| \|f_n\| < \infty$ and $\varphi(A) = \sum_n f_n(x_n \cdot A)$, $A \in M(\mathfrak{A})$. Hence, the functional $f: x \mapsto \sum_n f_n(\mathfrak{b}_T(x_n, y) \cdot x)$ belongs to $\mathfrak{X}_*$. Therefore, $g \in \mathfrak{X}_*$, because $g(x) = \sum_n f_n(x_n \cdot \mathfrak{a}_T(x, y)) = f(x)$.

In our former considerations we have already used the fact that each self-dual Hilbert module $(\mathfrak{H}, \langle \cdot \mid \cdot \rangle)$ over a W^*-algebra $\mathfrak{A}$ is a dual space. Hence, $(\mathfrak{H}, (\cdot, \cdot, \cdot), \|\cdot\|_{\mathfrak{H}})$ is a ternary W^*-ring, $(\cdot, \cdot, \cdot)$ given by $(x, y, z) = x \cdot \langle z \mid y \rangle$. Now, Corollary 4.11 implies that the canonical predual $\mathfrak{H}_c \otimes_{\mathfrak{A}} \mathfrak{A}_*$ coincides with $\mathfrak{H}_c \otimes \mathfrak{A}_*$ and is uniquely determined up to isomorphism. Our final example includes the converse of Theorem 2.8.

4.12. Example. Let $\mathfrak{R} \subset \mathscr{L}(H, K)$ be a weakly closed TRO. Then $(\mathfrak{R}, \langle \cdot \mid \cdot \rangle)$ is a self-dual Hilbert $\{\mathfrak{R}\}_H$-module, where $\langle A \mid B \rangle = B^*A$. In particular, $(\mathfrak{R}, (\cdot, \cdot, \cdot), \|\cdot\|)$ is a ternary W^*-ring with $(\cdot, \cdot, \cdot)$ given by $(A, B, C) = AB^*C$ and with unique predual $\mathfrak{R}_c \otimes (\{\mathfrak{R}\}_H)_*$.

Proof. To show self-duality, let τ: $\mathfrak{R} \to \{\mathfrak{R}\}_H$ be a bounded $\{\mathfrak{R}\}_H$-module homomorphism. Since $(A, f) \mapsto f(\tau(A)^*)$ is continuous on $\mathfrak{R}_c \times (\{\mathfrak{R}\}_H)_*$ there exists an η in the dual space of the projective tensor product $\mathfrak{R}_c \hat{\otimes}_\gamma (\{\mathfrak{R}\}_H)_*$ satisfying $\eta(A \otimes f) = f(\tau(A)^*)$, $A \in \mathfrak{R}$. Since

$$\eta((AX) \otimes f - A \otimes (f \cdot X)) = f(\tau(AX)^*) - f(X^*\tau(A)^*) = 0$$

we may assume that $\eta \in \mathfrak{R}$ which in turn yields

$$f(\tau(A)^*) = \eta(A \otimes f) = f(A^*\eta), \qquad A \in \mathfrak{R}, \quad f \in (\{\mathfrak{R}\}_H)_*$$

and therefore, $\tau = \langle \cdot \mid \eta \rangle$.

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